Problem 1

a)
The two spheres are connected, therefore they have the same potential:

$$V_1 = V_2$$

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 b_1} \qquad V_2 = \frac{Q_2}{4\pi\varepsilon_0 b_2}$$

$$\begin{cases} Q_1 + Q_2 = Q \\ \dfrac{Q_1}{4\pi\varepsilon_0 b_1} = \dfrac{Q_2}{4\pi\varepsilon_0 b_2} \end{cases}$$

$$\Rightarrow \quad Q_1 + \frac{b_2}{b_1} Q_1 = Q \Rightarrow \left(1 + \frac{b_2}{b_1}\right) Q_1 = Q$$

$$\Rightarrow Q_1 = \frac{b_1}{b_1+b_2} Q$$

$$Q_2 = \frac{b_2}{b_1+b_2} Q$$

b) According to Gauss law:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

The E field on the surface of the first sphere is:

$$E_{R1} = \frac{Q_1}{4\pi\varepsilon_0 b_1^2} = \frac{b_1}{b_1+b_2} \cdot \frac{Q}{4\pi\varepsilon_0 b_1^2} = \frac{Q}{4\pi\varepsilon_0 b_1(b_1+b_2)}$$

The E field on the surface of the second sphere is:

$$E_{R2} = \frac{Q_2}{4\pi\varepsilon_0 b_2^2} = \frac{Q}{4\pi\varepsilon_0 b_2(b_1+b_2)}$$

Problem 2

Current $I$ is uniformly distributed within the cylindrical shell, then the current density becomes:

$$J = \frac{I}{S} = \frac{I}{\pi(b^2-a^2)}$$

According to Ampere's current law:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu I_0$$

$$B_\theta \cdot 2\pi r = \mu I_0$$

$$B_\theta = \frac{\mu I_0}{2\pi r}$$

When $r \leq a$

$$I_0 = 0 \quad \Rightarrow \quad B_\theta = 0$$

When $a < r < b$

$$I_0 = J \cdot \pi(r^2-a^2) = \frac{I}{\pi(b^2-a^2)} \cdot (r^2-a^2)\pi = \frac{I(r^2-a^2)}{b^2-a^2}$$

$$B_\theta = \frac{\mu I(r^2-a^2)}{b^2-a^2} \cdot \frac{1}{2\pi r}$$

When $r \geq b$

$$I_0 = I \qquad B_\theta = \frac{\mu_0 I}{2\pi r}$$

Problem 3

a) when $U = 220\,V$

$I_c = 0$

$I_R = I_L = \frac{220V}{20\Omega} = 11A$

b) $U = 220\sqrt{2}\sin(100\pi t)$ $\quad \bar{U} = 220\sqrt{2}\, e^{j100\pi t}$.

$\bar{Z}_c = \frac{1}{j\omega C}$

$\bar{I}_c = \frac{\bar{U}}{\bar{Z}_c} = 220\sqrt{2}\, e^{j100\pi t} \cdot j100\pi \cdot 100 \cdot 10^{-6} = 2.2\sqrt{2}\pi\, e^{j(100\pi t + \frac{\pi}{2})}$

$= 6.9115\sqrt{2}\, e^{j(100\pi t + \frac{\pi}{2})}$

$\bar{I}_R = \bar{I}_L = \frac{\bar{U}}{R + j\omega L} = \bar{U}\,\frac{R - j\omega L}{R^2 + (\omega L)^2} = \bar{U}\,\frac{\sqrt{R^2 + (\omega L)^2}}{R^2 + (\omega L)^2}\, e^{-j\tan^{-1}\frac{\omega L}{R}}$

$= 220\sqrt{2}\, e^{j100\pi t} \cdot \frac{\sqrt{20^2 + (100\pi \cdot 50 \cdot 10^{-3})^2}}{20^2 + (100\pi \cdot 50 \cdot 10^{-3})^2} \cdot$

$e^{-j\tan^{-1}\frac{100\pi \cdot 50 \cdot 10^{-3}}{20}}$

$= 8.6508\sqrt{2}\, e^{j100\pi t}\, e^{-j\tan^{-1} 0.7854}$

The effective current:

$I_c^e = 6.9115\,A$. $\quad I_R^e = I_L^e = 8.6508\,A$

Problem 4

The operational amplifier is ideal, therefore we have:

$$\begin{cases} \bar{I}_1(R_2+R_3)+\bar{U}_{ut}=0 & ① \\ \bar{I}_2\frac{1}{j\omega C_1}+\bar{I}_2R_1-\bar{U}_{in}=0 & ② \\ \bar{I}_1R_2+\bar{I}_2R_1=0 & ③ \end{cases}$$

From ③ $\quad \bar{I}_2 = \dfrac{-R_2}{R_1}\bar{I}_1$

Insert into ②

$$-\frac{R_2}{R_1}\bar{I}_1\left(R_1+\frac{1}{j\omega c}\right)-\bar{U}_{in}=0$$

$$\bar{I}_1=\frac{-\bar{U}_{in}R_1}{R_2\left(R_1+\frac{1}{j\omega c}\right)}$$

From ① $\quad \bar{I}_1 = \dfrac{-\bar{U}_{ut}}{R_2+R_3}$

$$\Rightarrow \quad \frac{-\bar{U}_{in}R_1}{R_2\left(R_1+\frac{1}{j\omega c}\right)}=\frac{-\bar{U}_{ut}}{R_2+R_3}$$

$$\Rightarrow \quad H(j\omega)=\frac{\bar{U}_{ut}}{\bar{U}_{in}}=\frac{R_1(R_2+R_3)}{R_2\left(R_1+\frac{1}{j\omega c}\right)}=\frac{j\omega CR_1(R_2+R_3)}{R_2(1+j\omega R_1C)}$$

Problem 5

a)

A: 1X5

B: 3X3

C: 1X9

D: 5X5

E: 9X9

b) $g(x,y) = f(x,y) * h(x,y)$

$$= \sum_{-M}^{M} \sum_{-N}^{N} f(a,b)\, h(x-a, y-b)$$

$$g(-1,-1) = f(-1,-1)\,h(0,0) + f(-1,0)\,h(0,-1) + f(-1,1)\,h(0,-2)$$
$$+ f(0,-1)\,h(-1,0) + f(0,0)\,h(-1,-1) + f(0,1)\,h(-1,-2)$$
$$+ f(1,-1)\,h(-2,0) + f(1,0)\,h(-2,-1) + f(1,1)\,h(-2,-2)$$

$$= 0\cdot6 + 1\cdot2 + 0 + 1\cdot2 + 4\cdot1 + 0 + 0 + 0 + 0$$

$$= 8$$

$g(-1,0) = 16$

$g(0,0) = 32$

$$g(x,y) = \begin{bmatrix} 8 & 16 & 8 \\ 16 & 32 & 16 \\ 8 & 16 & 8 \end{bmatrix}$$