Lecture 9. Bayesian Inference - updating priors[1]

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[1] Bayesian statistics is a general methodology to analyse and draw conclusions from data.

$$P = P(\text{accidents happen in period } t) = 1 - \mathrm{e}^{-\lambda_A P(B)\, t} \approx \lambda_A P(B)\, t,$$

if probability P is small. Hence Two problems of interest in risk analysis:

- The first one will deal with the estimation of a probability $p_B = \mathrm{P}(B)$, say, of some event B, for example the probability of failure of some system. In figure $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$
- The second one is estimation of the probability that at least once an event A occurs in a time period of length t. The problem reduces itself to estimation of the intensity λ_A of A.

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The parameters p_B and λ_A are unknown.

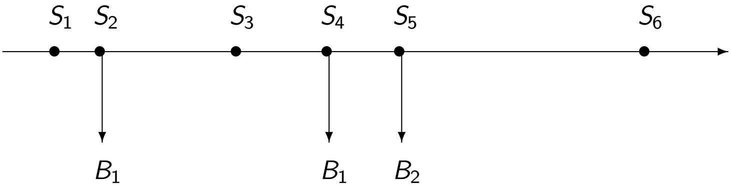


Figure: Events A at times S_i with related scenarios B_i.

Odds for parameters

Let θ denote the unknown value of p_B, λ_A or any other quantity.

Introduce odds q_θ, which for any pair θ_1, θ_2 represents our belief which of θ_1 or θ_2 is more likely to be the unknown value of θ, *i.e.* $q_{\theta_1} : q_{\theta_2}$ are odds for the alternatives $A_1 =$ "$\theta = \theta_1$" against $A_2 =$ "$\theta = \theta_2$".

We require that q_θ integrates to one and hence $f(\theta) = q_\theta$ is a probability density function representing our belief about the value of θ. The random variable Θ having the pdf serves as a mathematical model for uncertainty in the value of θ.

Prior odds - posterior ods

Let θ be the unknown parameter ($\theta = p_B$, $\theta = \lambda_A$), while Θ denotes any of the variables P or Λ. Since θ is unknown, it is seen as a value taken by a random variable Θ with pdf $f(\theta)$.

If $f(\theta)$ is chosen on basis of experience without including observations of outcomes of an experiment then the density $f(\theta)$ is called a *prior density* and denoted by $f^{\text{prior}}(\theta)$.

Since our knowledge may change with time (especially if we observe some outcomes of the experiment) influencing our opinions about the values of parameter θ. This leads to new odds - density $f(\theta)$. The modified density $f(\theta)$ will be called the *posterior density* and denoted by $f^{\text{post}}(\theta)$.

The method to update $f(\theta)$ is

$$f^{\text{post}}(\theta) = cL(\theta)\, f^{\text{prior}}(\theta)$$

How to find likelihood function $L(\theta)$ will be discussed later on.

Predictive probability

Suppose $f(p)$ has been selected and denote by P a random variable having pdf $f(p)$. A plot of $f(p)$ is an illustrative measure of how likely the different values of p_B are.

If only one value of the probability is needed, the Bayesian methodology proposes to use the so-called **predictive probability** which is simply the mean of P:

$$\mathsf{P}^{\mathsf{pred}}(B) = \mathsf{E}[P] = \int p f(p) \, \mathrm{d}p.$$

The predictive probability measures the likelihood that B occurs in future. It combines two sources of uncertainty: the unpredictability whether B will be true in a future accident and the uncertainty in the value of probability p_B.

Example 6.1

$$P(A \cap B) = P(\text{accidents in period } t) = 1 - \mathrm{e}^{-\lambda_A P(B)\, t} \approx \lambda_A P(B)\, t,$$

if probability $P(A \cap B)$ is small.

The predictive probabilities

$$\begin{aligned}
\mathrm{P}^{\mathrm{pred}}(A) &= \mathrm{E}[P(A)] = \int (1 - \exp(-\lambda\, t)) f_\Lambda(\lambda)\, \mathrm{d}\lambda \\
&\approx \int t\lambda f_\Lambda(\lambda)\, \mathrm{d}\lambda = t\mathrm{E}[\Lambda].^{2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}^{\mathrm{pred}}(A \cap B) &= \int (1 - \exp(-p\lambda\, t)) f_\Lambda(\lambda) f_P(p)\, \mathrm{d}\lambda\, \mathrm{d}p \\
&\approx \int t\, p\lambda f_\Lambda(\lambda) f_P(p)\, \mathrm{d}\lambda\, \mathrm{d}p = t\mathrm{E}[\Lambda]\mathrm{E}[P].
\end{aligned}$$

Example 6.2

[2] For small x, $1 - \exp(-x) \approx x$.

Credibility intervals:

- In the Bayessian approach the lack of knowledge of parameter value θ is described using the probability densities $f(\theta)$ (odds). Random variable Θ having the pdf $f(\theta)$ models our knowledge about θ.

- The initial knowledge is described using $f^{\text{prior}}(\theta)$ density and as the data are gathered it is updated

$$f^{\text{post}}(\theta) = c\, L(\theta) f^{\text{prior}}(\theta).$$

- The pdf $f^{\text{post}}(\theta)$ summarizes our knowledge about θ. However if one value of for the parameter is needed then

$$\theta^{\text{predictive}} = \mathsf{E}[\Theta] = \int \theta f^{\text{post}}(\theta)\, d\theta.$$

- If one wishes to describe the variability of θ by means of an interval then the so called credibility interval can be computed

$$[\,\theta^{\text{post}}_{1-\alpha/2},\ \theta^{\text{post}}_{\alpha/2}\,]$$

Gamma-priors:

Conjugated priors are families of pdf for Θ which are particularly convenient for recursive updating procedures, *i.e.* when new observations arrive at different time instants. We will use three families of conjugated priors:

Gamma pdf:

$\Theta \in \mathsf{Gamma}(a, b), \quad a, b > 0, \quad$ if

$$f(\theta) = c\,\theta^{a-1}\mathrm{e}^{-b\theta}, \quad \theta \geq 0, \quad c = \frac{b^a}{\Gamma(a)}.$$

The expectation, variance and coefficient of variation for $\Theta \in \mathsf{Gamma}(a, b)$ are given by

$$\mathsf{E}[\Theta] = \frac{a}{b}, \qquad \mathsf{V}[\Theta] = \frac{a}{b^2}, \qquad \mathsf{R}[\Theta] = \frac{1}{\sqrt{a}}.$$

Updating Gamma priors:

The Gamma priors are conjugated priors for the problem of estimating the intensity in a Poisson stream of events A. If one has observed that in time $\widetilde{t}$ there were k events reported and if the prior density $f^{\text{prior}}(\theta) \in Gamma(a, b)$, then

$$f^{\text{post}}(\theta) \in \text{Gamma}(\widetilde{a}, \widetilde{b}), \qquad \widetilde{a} = a + k, \quad \widetilde{b} = b + \widetilde{t}.$$

Further, the predictive probability of at least one event A during a period of length t is given by

$$\text{P}^{\text{pred}}(A) \approx t\text{E}[\Theta] = t\,\frac{\widetilde{a}}{\widetilde{b}}$$

In Example 6.2 the $f^{\text{prior}}(\theta)$ was exponential with mean $1/30$ [days^{-1}]. This is Gamma(1,30) pdf. Suppose that in 10 days we have not observed any accidents then posteriori density $f^{\text{post}}(\theta)$ is Gamma(1,40). Hence

$$\text{P}^{\text{pred}}(A) \approx \frac{t}{40}.$$

Conjugated Beta-priors:

Beta probability-density function (pdf):

$\Theta \in \mathrm{Beta}(a, b), \quad a, b > 0$, if

$$f(\theta) = c\,\theta^{a-1}(1-\theta)^{b-1}, \quad 0 \le \theta \le 1, \quad c = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

The expectation and variance of $\Theta \in \mathrm{Beta}(a, b)$ are given by

$$\mathrm{E}[\Theta] = p, \qquad \mathrm{V}[\Theta] = \frac{p(1-p)}{a+b+1},$$

where $p = a/(a+b)$. Furthermore, the coefficient of variation

$$\mathrm{R}(\Theta) = \frac{1}{\sqrt{a+b+1}}\sqrt{\frac{1-p}{p}}.$$

Updating Beta-priors:

The Beta priors are conjugated priors for the problem of estimating the probability $p_B = \mathsf{P}(B)$.

Let $\theta = p_B$. If one has observed that in n trials (results of experiments), the statement B was true k times and if the prior density $f^{prior}(\theta) \in Beta(a, b)$ then

$$f^{\text{post}}(\theta) \in \text{Beta}(\widetilde{a}, \widetilde{b}), \qquad \widetilde{a} = a + k, \quad \widetilde{b} = b + n - k.$$

$$\mathsf{P}^{\text{pred}}(B) = \int_0^1 \theta f^{\text{post}}(\theta)\, d\theta = \frac{\widetilde{a}}{\widetilde{a} + \widetilde{b}}.$$

Consider example of treatment of waste water. Let p be the probability that water is sufficiently cleaned after a week of treatment. If we have no knowledge about p we could use the uniform priors. It is easy to see that it is Beta(1,1) pdf.
Suppose that 3 times water was well cleaned and 2 times not. This information gives the posterior density Beta(4,3) and the predictive probability that water is cleaned in one week is 4/7.

Conjugated Dirichlet-priors:

Dirichlet's pdf:

$\Theta = (\Theta_1, \Theta_2) \in \text{Dirichlet}(\mathbf{a})$, $\mathbf{a} = (a_1, a_2, a_3)$, $a_i > 0$, if

$$f(\theta_1, \theta_2) = c\,\theta_1^{a_1-1}\theta_2^{a_2-1}(1-\theta_1-\theta_2)^{a_3-1}, \quad \theta_i > 0, \theta_1 + \theta_2 < 1,$$

where $c = \frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$. Let $a_0 = a_1 + a_2 + a_3$; then

$$\mathrm{E}[\Theta_i] = \frac{a_i}{a_0}, \quad \mathrm{V}[\Theta_i] = \frac{a_i(a_0 - a_i)}{a_0^2(a_0+1)}, \quad i = 1, 2.$$

Furthermore the marginal probabilities are Beta distributed, *viz.*

$$\Theta_i \in \text{Beta}(a_i, a_0 - a_i), \quad i = 1, 2.$$

Updating Dirichlet's priors.

The Dirichlet priors are conjugated priors for the problem of estimating the probabilities $p_i = \mathsf{P}(B_i)$, $i = 1, 2, 3$, B_i are disjoint, $p_1 + p_2 + p_3 = 1$.

Let $\theta_i = p_i$. If one has observed that the statement B_i was true k_i times in n trials and the prior density $f^{prior}(\theta_1, \theta_2) \in Dirichlet\,(\mathbf{a})$,

$$f^{\text{post}}(\theta_1, \theta_2) \in \text{Dirichlet}\,(\widetilde{\mathbf{a}}), \quad \widetilde{\mathbf{a}} = (a_1 + k_1,\, a_2 + k_2,\, a_3 + k_3),$$

where $k_3 = n - k_1 - k_2$. Further

$$\mathsf{P}^{\text{pred}}(B_i) = \mathsf{E}[\Theta_i] = \frac{\widetilde{a}_i}{\widetilde{a}_1 + \widetilde{a}_2 + \widetilde{a}_3}.$$

Let B_1="player A wins", B_2="player B wins" (there is possibility of draw). If we do not know strength of players we could use uniform priors which corresponds to Dirichlet(1,1,1) pdf. Now we observed that in two matches A won twice, hence the posteriori density is Dirichlet(3,1,1) and the predictive probability that A wins the next match is then $3/5$.

Posterior pdf for large number of observations.

If $f^{\text{prior}}(\theta_0) > 0$ then $\Theta \in \text{AsN}(\theta^*, (\sigma^*_{\mathcal{E}})^2)$ as $n \to \infty$, where θ^* is the ML estimate of θ_0 and $\sigma^*_{\mathcal{E}} = 1/\sqrt{-\ddot{l}(\theta^*)}$.

It means that

$$f^{\text{post}}(\theta) \approx c\ \exp\big(\frac{1}{2}\ddot{l}(\theta^*)(\theta-\theta^*)^2\big) = c\ \exp\big(-\frac{1}{2}\big((\theta-\theta^*)^2/(\sigma^*_{\mathcal{E}})^2\big)\big).$$

Sketch of proof:

$$l(\theta) \approx l(\theta^*) + \dot{l}(\theta^*)(\theta-\theta^*) + \frac{1}{2}\ddot{l}(\theta^*)(\theta-\theta^*)^2.$$

Now likelihood function $L(\theta) = \mathrm{e}^{l(\theta)}$ and $\dot{l}(\theta^*) = 0$, thus

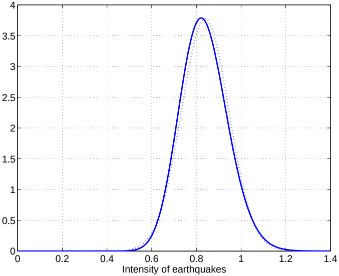
$$\begin{aligned} L(\theta) &\approx \exp\Big(l(\theta^*) + \dot{l}(\theta^*)(\theta-\theta^*) + \frac{1}{2}\ddot{l}(\theta^*)(\theta-\theta^*)^2\Big) \\ &= c\ \exp\big(\frac{1}{2}\ddot{l}(\theta^*)(\theta-\theta^*)^2\big). \end{aligned}$$

Example earthquake data:

We have demonstrated that time between earthquakes is Exp(a). Here it is more convenient to use parameter $\theta = 1/a$, i.e. the intensity of earthquakes. The ML estimate $\theta^* = 1/\bar{x}$ and $\ddot{l}(\theta) = -n/\theta^2$. Since $\bar{x} = 437.2$ days we have that $\theta^* = 364/437.2 = 0.8395$ years^{-1}, while

$$(\sigma_{\mathcal{E}}^*)^2 = \frac{(\theta^*)^2}{n} = 0.0112.$$

Consequently $\Theta^* \approx \mathrm{N}(0.8395, 0.0112)$. This can be used to give approx. confidence interval for θ or $p = \mathrm{P}(T > 4.1) = \exp(-4.1\,\theta)$.



Let use non-informative priors $f^{\text{prior}}(\theta) = 1/\theta$ then the gamma posterior density has parameters $a = 62$ and $b = (437.2/365) \cdot 62 = 74.26$; $f^{\text{post}}(\theta) \in \text{Gamma}(62, 74.26)$ (solid line): Asymptotic normal posterior pdf N(0.8395, 0.0112) (dotted line).

Transport of nuclear fuel waste

Spent nuclear fuel is transported by railroad. From historical data, one knows that there were 4 000 transports without a single release of radioactive material. Since fuel waste is highly dangerous, one has discussed the possibility of constructing a special (very safe and expensive) train to transport the spent fuel.

One problem was the definition of an acceptable risk p^{acc} for an accident, i.e. one wishes the probability of an accident θ, say, to be smaller than p^{acc}. Since θ is unknown and uncertainty of its value is modelled by a random variable Θ the issue is to check, on basis of available data and experience, whether the predictive probability $\mathsf{P}(\Theta < p^{\text{acc}})$ is high.

A number between 10^{-8} and 10^{-10} was first proposed for p^{acc}, i.e. the average waiting time for an accident is 10^{8} to 10^{10} transports. In such a scale the experienced 4000 safe transports looks clearly negligible and hence the conclusion was: if one wishes to transport the waste with the required reliability, one needs to develop transport systems with maximum reliability.

How the information about 4 000 problem free transports affects our believes about risk for accidents. Suppose that accidents happen independently with probability θ. Then[3]

$$\mathsf{P}(\text{“No accidents for 4 000 transports”} \mid \Theta = \theta) = (1-\theta)^{4000} \approx \mathrm{e}^{-4000\,\theta},$$

and the posterior density $f^{\text{post}}(\theta) = cf^{\text{prior}}(\theta)\mathrm{e}^{-4000\,\theta}$ will be close to zero for any reasonable choice of the prior density and $\theta > 10^{-3}$. This agrees with the conclusion of Kaplan and Garrick that the information of 4 000 release-free transport is quite informative:

> *“The experience of* 4 000 *release-free shipments is not sufficient to distinguish between release frequencies of* 10^{-5} *or less. However, it is sufficient to substantially reduce our belief that the frequency is on the order of* 10^{-4} *and virtually demolish any belief that the frequency could be* 10^{-3} *or greater”.*

If we assume that the required safety is $p = 10^{-8}$, then the information of 4 000 accident-free transports is insignificant; on the other hand, the required safety may never be checked.

[3] Here we use that for small θ, $\mathrm{e}^{-\theta} \approx 1-\theta$. In addition $\lim_{n\to\infty}\left(1-\frac{a}{n}\right)^{n} = \mathrm{e}^{-a}$.