Lecture 10. Failure Probabilities and Safety Indexes

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Safety analysis - General setup:

An alternative method to compute risk, here the probability of at least one accident in one year, is to identify streams of events A_i which, if followed by a suitable scenario B_i, leads to the accident. Then the risk for the accident is approximately measured by $\sum \lambda_{A_i} \mathrm{P}(B_i)$[1] where the intensities of the streams of A_i, λ_{A_i}, all have units [year^{-1}].

An important assumption is that the streams of initiation events are independent and much more frequent than the occurrences of studied accidents. Hence these can be estimated from historical records.

What remains is computation of probabilities $\mathrm{P}(B_i)$.

We consider cases when the scenario B describes the ways systems can fail, or generally, some risk-reduction measures fail to work as planned.

In safety of engineering structures, B is often written in a form that a function of uncertain values (random variables) exceeds some critical level u^{crt}

$$B = \text{“}g(X_1, X_2, \ldots, X_n) > u^{\mathrm{crt}}\text{”}$$

[1] $1 - \exp(-x) \approx x$

Failure probability:

Some of the variables X_i may describe uncertainty in parameters, model, etc. while others genuine random variability of the environment. One thus mixes the variables X with distributions interpreted in the frequentist's way with variables having subjectively chosen distributions. Hence the interpretation of what the **failure probability**

$$P_{\mathrm{f}} = \mathsf{P}(B) = \mathsf{P}(g(X_1, X_2, \ldots, X_n) > u^{\mathsf{crt}})$$

means is difficult and depends on properties of the analysed scenario.

It is convenient to find a function h such that

$$B \;=\; \text{"}\, h(X_1, X_2, \ldots, X_n) \le 0\text{"}.$$

Then, with $Z = h(X_1, X_2, \ldots, X_n)$, the failure probability $P_{\mathrm{f}} = F_Z(0)$.[2]
One might think that it is a simple matter to find the failure probability P_{f}, since only the distribution of a single variable Z needs to be found.

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[2] Often $h(X_1, X_2, \ldots, X_n) = u^{\mathsf{crt}} - g(X_1, X_2, \ldots, X_n)$. Note that h is not uniquely defined.

Example - summing many small contributions:

By Hooke's law, the elongation ϵ of a fibre is proportional to the force F, that is, $\epsilon = F/K$ or $F = K\epsilon$. Here K, called Young's modulus, is uncertain and modelled as a rv. with mean m and variance σ^2.

Consider a wire containing 1000 fibres with individual independent values of Young's modulus K_i. A safety criterion is given by $\epsilon \le \epsilon_0$. With $F = \epsilon \sum K_i$ we can write

$$P_{\mathrm{f}} = \mathrm{P}\Big(\frac{F}{\sum K_i} > \epsilon_0\Big) = \mathrm{P}(\epsilon_0 \sum K_i - F < 0).$$

Hence, in this example, we have

$$h(K_1, \ldots, K_{1000}, F) = \epsilon_0 \sum K_i - F$$

which is a linear function of K_i and F.[3]

[3] Here, F is an external force (load) while $\sum K_i$ is the material strength.

Assume $F \in \mathrm{N}(m_F, \sigma_F^2)$ is independent of K_i ($\mathrm{E}[K_i] = m$, $\mathrm{V}[K_i] = \sigma^2$).

By the central limit theorem, $\sum K_i$ is approximately $\mathrm{N}(1000m, 1000\sigma^2)$. Hence $Z = \epsilon_0 \sum K_i - F$, is the difference of two independent normal variables. Since

sum of independent normally distributed variables has normal distribution.[4]

hence $Z \in \mathrm{N}(m_Z, \sigma_Z^2)$ where $m_Z = 1000m\epsilon_0 - m_F$, $\sigma_Z^2 = 1000\,\epsilon_0^2\sigma^2 + \sigma_F^2$.

Consequently $P_{\mathrm{f}} = \mathrm{P}(Z < 0) = \Phi\left(\frac{-m_Z}{\sigma_Z}\right)$.

Bigger the fraction $\beta_{\mathrm{C}} = \frac{m_Z}{\sigma_Z}$ lower the probability of failure.

[4]Sum of jointly normally distributed variables (can be dependent) is normally distributed too.

Some results for sums:

- If $X_1, \ldots, X_n$ are independent normally distributed, i.e. $X_i \in \mathrm{N}(m_i, \sigma_i^2)$, then their sum Z is normally distributed too, i.e. $Z \in \mathrm{N}(m, \sigma^2)$, where

$$m = m_1 + \cdots + m_n, \qquad \sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2.$$

- For independent Gamma distributed random variables $X_1, X_2, \ldots, X_n$, where $X_i \in \mathrm{Gamma}(a_i, b)$, $i = 1, \ldots, n$, one can show that

$$\sum_{i=1}^{n} X_i \in \mathrm{Gamma}(a_1 + a_2 + \cdots + a_n,\ b).$$

- Sum of independent Poisson variables, $K_i \in \mathrm{Po}(m_i)$, $i = 1, \ldots, n$, is again Poisson distributed:

$$\sum_{i=1}^{n} K_i \in \mathrm{Po}(m_1 + \cdots + m_n).$$

  Recall the more general results of superposition and decomposition of Poisson processes

The weakest-link principle:

The principle means that the strength of a structure is equal to the strength of its weakest part. For a chain "failure" occurs if minimum of strengths of chain components is below a critical level u^{crt}:

$$\min(X_1, \ldots, X_n) \leq u^{\text{crt}}.$$

If X_i are independent with distributions F_i, then

$$\begin{aligned} P(\min(X_1, \ldots, X_n) \leq u^{\text{crt}}) &= 1 - P(\min(X_1, \ldots, X_n) > u^{\text{crt}}) \\ &= 1 - P(X_1 > u^{\text{crt}}, \ldots, X_n > u^{\text{crt}}) \\ &= 1 - (1 - F_1(u^{\text{crt}})) \cdot \ldots \cdot (1 - F_n(u^{\text{crt}})). \end{aligned}$$

The computations are particularly simple if X_i are iid Weibull distributed then the cdf of $X = \min(X_1, X_2, \ldots, X_k)$ is

$$P(X \leq x) = 1 - (1 - (1 - e^{-(x/a)^c}))^k = 1 - e^{-k(x/a)^c} = 1 - e^{-(x/a_k)^c},$$

that is, a Weibull distribution with a new scale parameter $a_k = a/k^{1/c}$.[5]

[5] The change of scale parameter due to minimum formation is called *size effect* (larger objects are weaker).

Example: Strength of a wire

Experiments have been performed with 5 cm long wires. Estimated average strength was 200 kg and coefficient of variation 0.20. From experience, one knows that such wires have Weibull distributed strengths.

For Weibull cdf $F(x) = 1 - e^{-(x/a)^c}$, $x > 0$,
$\mathsf{R}(X) = \frac{\sqrt{\Gamma(1+2/c) - \Gamma^2(1+1/c)}}{\Gamma(1+1/c)}$.

c	$\Gamma(1+1/c)$	$\mathsf{R}(X)$
1.00	1.0000	1.0000
2.00	0.8862	0.5227
2.10	0.8857	0.5003
2.70	0.8893	0.3994
3.00	0.8930	0.3634
3.68	0.9023	0.3025
4.00	0.9064	0.2805
5.00	0.9182	0.2291
5.79	0.9259	0.2002
8.00	0.9417	0.1484
10.00	0.9514	0.1203
12.10	0.9586	0.1004
20.00	0.9735	0.0620
21.80	0.9758	0.0570
50.00	0.9888	0.0253
128.00	0.9956	0.0100

The table gives $c = 5.79$ and $\Gamma(1 + 1/c) = 0.9259$. Next using the relation $a = \mathsf{E}[X]/\Gamma(1 + 1/c)$ one gets $a = 200/0.9259 = 216.01$.

We now consider strength of a 5 meters long wire. It is 100 times longer than the tested wires and hence its strength is Weibull distributed with $c = 5.79$ and $a = 216.01/100^{1/c} = 97.51$. In average the 5 meter long wires are 2.22 weaker than the 5 cm long test specimens.

Now we can calculate the probability that a wire of length 5 m will have a strength less than 50 kg,

$$P(X \leq 50) = 1 - \mathrm{e}^{-(50/97.51)^{5.79}} = 0.021.$$

For the 5 cm long test specimens

$$P(X \leq 50) = 1 - \mathrm{e}^{-(50/216)^{5.79}} = 0.00021,$$

i.e. 100 times smaller. Not surprising since $1 - \exp(-x) \approx x$ for small x values.

Multiplicative models:

Assume that January 2009, one has invested K SEK in a stock portfolio and one wonders what its value will be in year 2020. Denote the value of the portfolio in year 2020 by Z and let X_i be factors by which this value changed during a year $2009 + i$, $i = 0, 1, \ldots, 11$. Obviously the value is given by

$$Z = K \cdot X_0 \cdot X_1 \cdot \ldots \cdot X_{11}.$$

Here "failure" is subjective and depends on our expectations, e.g. "failure" can be that we lost money, i.e. $Z < K$.

In order to estimate the risk (probability) for failure, one needs to model the properties of X_i. As we know factors X_i are either independent nor have the same distribution.[6] For simplicity suppose that X_i are iid, then employing logarithmic transformation

$$\ln Z = \ln K + \ln X_1 + \cdots + \ln X_n,$$

Now if n is large the Central Limit Theorem tells us that $\ln Z$ is approximatively normally distributed.

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[6] The so called theory of *time series* is often used to model variability of X_i.

Lognormal rv. :

A variable Z such that $\ln Z \in \mathrm{N}(m, \sigma^2)$ is called a **lognormal variable**.

Using the distribution Φ of a $\mathrm{N}(0,1)$ variable we have that

$$F_Z(z) = \mathrm{P}(Z \le z) = \mathrm{P}(\ln Z \le \ln z) = \Phi\left(\frac{\ln z - m}{\sigma}\right).$$

In can be shown that

$$\begin{aligned}
\mathrm{E}[Z] &= \mathrm{e}^{m+\sigma^2/2}, \\
\mathrm{V}[Z] &= \mathrm{e}^{2m} \cdot (\mathrm{e}^{2\sigma^2} - \mathrm{e}^{\sigma^2}), \\
\mathrm{D}[Z] &= \mathrm{e}^{m}\sqrt{\mathrm{e}^{2\sigma^2} - \mathrm{e}^{\sigma^2}} = \mathrm{e}^{m+\sigma^2/2} \cdot \sqrt{\mathrm{e}^{\sigma^2} - 1}.
\end{aligned}$$

Please study applications of log-normally distributed variables given in the course book.

Safety Indexes:

A safety index is used in risk analysis as a measure of safety which is high when the probability of failure P_f is low. This measure is a more crude tool than the probability, and is used when the uncertainty in P_f is too large or when there is not sufficient information to compute P_f.

Consider the simplest case $Z = R - S$ and suppose that variables R and S are independent normally distributed, *i.e.* $R \in \mathrm{N}(m_R, \sigma_R^2)$, $S \in \mathrm{N}(m_S, \sigma_S^2)$. Then also $Z \in \mathrm{N}(m_Z, \sigma_Z^2)$, where $m_Z = m_R - m_S$ and $\sigma_Z = \sqrt{\sigma_R^2 + \sigma_S^2}$, and thus

$$P_\mathrm{f} = \mathrm{P}(Z < 0) = \Phi\big(\frac{0 - m_Z}{\sigma_Z}\big) = \Phi(-\beta_\mathrm{C}) = 1 - \Phi(\beta_\mathrm{C}),$$

where $\beta_\mathrm{C} = m_Z/\sigma_Z$ is called *Cornell's safety index*.

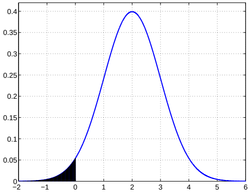

Illustration of safety index. Here: $\beta_\mathrm{C} = 2$.
Failure probability $P_\mathrm{f} = 1 - \Phi(2) = 0.023$
(area of shaded region).

Cornell - index

The index β_C gives the failure probabilities when Z is approximately normally distributed. Note that for any distribution of Z the Cornell's safety index $\beta_C = 4$ always means that the distance from the mean of Z to the unsafe region is 4 standard deviations. In quality control 6 standard deviations[7] are used lately, however in that case one is interested in fraction of components that do not meet specifications. In our case we do not consider mass production but long exposures times.

Even if in general $P_{\mathrm{f}} \neq 1 - \Phi(\beta_C)$ there exists, although very conservative, estimate

$$\mathsf{P}(\text{“System fails”}) = \mathsf{P}(Z < 0) \leq \frac{1}{1 + \beta_C^2}.$$

The Cornells index has some deficiencies and hence an improved version, called Hasofer-Lind index, is commonly used in reliability analysis. Since quite advanced computer software is needed for computation of β_{HL} it will not be discussed in details.

[7] Six Sigma is a registered service mark and trademark of Motorola, Inc. Motorola has reported over US$ 17 billion in savings from Six Sigma as of 2006.

Use of safety indexes in risk analysis

For β_{HL}, one has approximately that $P_{\mathrm{f}} \approx \Phi(-\beta_{\mathsf{HL}})$. Clearly, a higher value of the safety index implies lower risk for failure but also a more expensive structure. In order to propose the so-called **target safety index** one needs to consider both costs and consequences. Possible *classes of consequences* are:

Minor Consequences This means that risk to life, given a failure, is small to negligible and economic consequences are small or negligible (*e.g.* agricultural structures, silos, masts).

Moderate Consequences This means that risk to life, given a failure, is medium or economic consequences are considerable (*e.g.* office buildings, industrial buildings, apartment buildings).

Large Consequences This means that risk to life, given a failure, is high or that economic consequences are significant (*e.g.* main bridges, theatres, hospitals, high-rise buildings).

Obviously, the cost of risk prevention etc. also has to be considered, when we are choosing target reliability indexes ("target" means that one wishes to design the structures so that the safety index for a particular failure mode will have the target value). Here the so-called "ultimate limit states" are considered, which means failure modes of the structure — in everyday-language: that one can not use it anymore.

It is important to remember that the values of β_{HL} contain time information; it is a measure of safety for *one year*. Index $\beta_{\mathsf{HL}} = 3.7$ means that "nominal" return period for failure A, say, is 10^4 years. (Note that If you have 1000 independent streams of A then return period is only 10 years.)

Table 1: Safety index and consequences.

Relative cost of safety measure	Minor consequences of failure	Moderate consequences of failure	Large consequences of failure
Large	$\beta_{\mathrm{HL}} = 3.1$	$\beta_{\mathrm{HL}} = 3.3$	$\beta_{\mathrm{HL}} = 3.7$
Normal	$\beta_{\mathrm{HL}} = 3.7$	$\beta_{\mathrm{HL}} = 4.2$	$\beta_{\mathrm{HL}} = 4.4$
Small	$\beta_{\mathrm{HL}} = 4.2$	$\beta_{\mathrm{HL}} = 4.4$	$\beta_{\mathrm{HL}} = 4.7$

Computation of Cornell's index

- Although Cornell's index β_C has some deficiencies it is still an important measure of safety.
- Recall the setup: R_i are strength-, S_i the load-variables and $h(\cdot)$-function of strengthes and loads being negative when failure occurs. Let
$$Z = h(R_1, \dots, R_k, S_1, \dots, S_n),$$
and assume that $\mathsf{E}[Z] > 0$. Now $\beta_C = \mathsf{E}[Z]/\mathsf{V}[Z]^{1/2}$.
- Assume that only expected values and variances of the variables R_i and S_i are known. (We also assume that all strength and load variables are independent.) In order to compute β_C we need to find
$$\mathsf{E}[h(R_1, \dots, R_k, S_1, \dots, S_n)], \qquad \mathsf{V}[h(R_1, \dots, R_k, S_1, \dots, S_n)].$$
which often can only be done by means of some approximations. The main tools are the so-called *Gauss' formulae*.

Gauss' Approximations.

Let X be a random variable with $\mathsf{E}[X] = m$ and $\mathsf{V}[X] = \sigma^2$ then

$$\mathsf{E}[h(X)] \approx h(m) \quad \textit{and} \quad \mathsf{V}[h(X)] \approx (h'(m))^2\sigma^2.^8$$

Let X and Y be independent random variables with expectations m_X, m_Y, respectively. For a smooth function h the following approximations

$$\begin{aligned} \mathsf{E}[h(X, Y)] &\approx h(m_X, m_Y), \\ \mathsf{V}[h(X, Y)] &\approx \left[h_1(m_X, m_Y)\right]^2 \mathsf{V}[X] + \left[h_2(m_X, m_Y)\right]^2 \mathsf{V}[Y], \end{aligned}$$

where

$$h_1(x, y) = \frac{\partial}{\partial x} h(x, y), \qquad h_2(x, y) = \frac{\partial}{\partial y} h(x, y).$$

[8] Use Taylor's formula to approximate h around x_0 by a polynomial function $h(x) \approx h(x_0) + h'(x_0)(x - x_0)$. Choose "typical value" $x_0 = \mathsf{E}[X] = m$.

If X and Y are correlated then

$$\begin{aligned} \mathsf{E}[h(X,Y)] &\approx h(m_X, m_Y), \\ \mathsf{V}[h(X,Y)] &\approx \big[h_1(m_X, m_Y)\big]^2 \mathsf{V}[X] + \big[h_2(m_X, m_Y)\big]^2 \mathsf{V}[Y] \\ &\quad + 2h_1(m_X, m_Y)\, h_2(m_X, m_Y)\, \mathsf{Cov}[X, Y]. \end{aligned}$$

Extension to higher dimension then 2 is straightforward.

For independent strength and load variables Cornell's index can be approximately computed by the following formula

$$\beta_{\mathsf{C}} \approx \frac{h(m_{R_1}, \ldots, m_{R_k}, m_{S_1}, \ldots, m_{S_n})}{\left[\sum_{i=1}^{k+n} \big[\, h_i(m_{R_1}, \ldots, m_{R_k}, m_{S_1}, \ldots, m_{S_n}) \,\big]^2 \sigma_i^2\right]^{1/2}},$$

where σ_i^2 is the variance of the ith variable in the vector of loads and strengths $(R_1, \ldots, R_k, S_1, \ldots, S_n)$, while h_i denote the partial derivatives of the function h.

Example - displacement of a beam

Suppose that for a beam in a structure the vertical displacement U must be smaller than 1.5 mm. A formula from mechanics says that the vertical displacement of the midpoints is

$$U = \frac{PL^3}{48EI}.$$

Estimate a safety index, i.e. compute $\beta_C = E[Z]/V[Z]^{1/2}$, where $Z = 1.5 \cdot 10^{-3} - U$. Obviously

$$E[Z] = 1.5 \cdot 10^{-3} - E[U], \qquad V[Z] = V[U].$$ [9]

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[9] The data you find is; beam length $L = 3$ m; P is a random force applied at the midpoint $E[P] = 25\,000$ N and $D[P] = 5\,000$ N; the modulus of elasticity E of a randomly chosen beam has $E[E] = 2 \cdot 10^{11}$ Pa and $D[E] = 3 \cdot 10^{10}$ Pa; all beams share the same second moment of (cross-section) area $I = 1 \cdot 10^{-4}$ m^4. It seems reasonable to assume that P and E are uncorrelated.

Use of Gauss formulae

- Introducing $h(P, E) = \frac{PL^3}{48EI}$ we have

$$h_1(P, E) = \frac{\partial}{\partial P} h(P, E) = \frac{L^3}{48EI}, \quad h_2(P, E) = \frac{\partial}{\partial E} h(P, E) = -\frac{PL^3}{48E^2 I},$$

- Employing Gauss formulae

$$\begin{aligned} \mathrm{E}[U] &= \frac{\mathrm{E}[P]L^3}{48\mathrm{E}[E]I} = \frac{25\ 000 \cdot 3^3}{48 \cdot 2 \cdot 10^{11} \cdot 1 \cdot 10^{-4}} = 7.03 \cdot 10^{-4} \text{ m}, \\ \mathrm{V}[U] &= \mathrm{V}[P]\big[\, h_1(\mathrm{E}[P], \mathrm{E}[E])\,\big]^2 + \mathrm{V}[E]\big[\, h_2(\mathrm{E}[P], \mathrm{E}[E])\,\big]^2 = 1.11 \cdot 10^{-8} \end{aligned}$$

- Since $\mathrm{D}[U] = 1.06 \cdot 10^{-4}$ m and the Cornell's index[10]

$$\beta_C = (1.5 \cdot 10^{-3} - \mathrm{E}[U])/\mathrm{D}[U] = 7.52.$$

[10] $\mathrm{P}(Z < 0) \leq \frac{1}{1+\beta_C^2} = 0.017$