Föreläsning 3/9-13 fm

Komplexa tal

Gauss def. $\mathbb{C} = \{z = x + iy;\ x \in \mathbb{R}\ y \in \mathbb{R}\}$

i nytt tal, uppfyller $i^2 = -1$

$z = x + iy$, $w = u + iv$

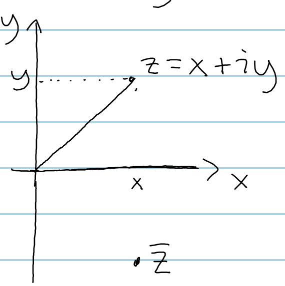
$z + w = u + x + i(y+v)$, $z \cdot w = xu - yv + i(xv + yu)$

Introduktion av i gör att ekvationen $z^2 = -1$ dvs $z^2 + 1 = 0$ lösbar!

Gör att alla ekvationer

$z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0$ lösbara!

$z = x + iy$ $\quad x = \operatorname{Re} z$, $y = \operatorname{Im} z$



$|z| = \sqrt{x^2 + y^2}$

$\bar{z} = x - iy$

$|z|^2 = \underbrace{(x+iy)(x-iy)}_{= z \cdot \bar{z}} = x^2 + y^2$ konjugatregeln

$z = a + ib$, $w = c + id$

$\bar{z} \cdot \bar{w} = (a - ib)(c - id) = ac - bd - i(bc + ad) = \overline{zw}$

$\Longrightarrow$ $\bar{z} \cdot \bar{w} = \overline{zw}$

Obs. $|zw| = |z||w|$

dvs $\sqrt{(xu - yv)^2 + (xv + yu)^2} =$

$= \sqrt{x^2 + y^2}\sqrt{u^2 + v^2}$

Bevis

$|zw|^2 = z\cdot w\cdot \overline{zw} = z\, w\, \bar{z}\, \bar{w} = z\bar{z}\, w\bar{w} = |z|^2 |w|^2$

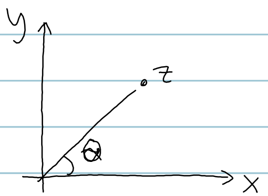
Division $z/w = ?$, $w \neq 0$

Först: $z = x + iy$, $w = u \in \mathbb{R}$

$\frac{z}{u} = \frac{x}{u} + \frac{iy}{u}$ OK.

Nu: $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$ OK.

Polär framställning



$|z|$ = längden

θ = vinkeln = $\arg(z)$

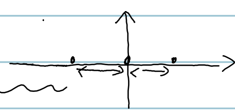
polär framställning

$|z| = r$, $\arg(z) = \theta$, $z = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

$|z - w|$ = avståndet från z till w

ex $|w-2| = |w+2|$



Multiplikation

$\underbrace{r(\cos\theta + i\sin\theta)}_{z}\ \underbrace{\rho(\cos\phi + i\sin\phi)}_{w} =$

$= r\rho[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi)] =$

$= r\rho[\cos(\theta+\phi) + i\sin(\theta+\phi)]$

$\because |zw| = |z||w|$, $\arg(zw) = \arg z + \arg w$

de Moivres sats

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\arg VL = n\theta = \arg HL)$

Obs. Om $\langle z, w\rangle = \{\text{skalärprodukten}\} =$

$= |z||w| \cdot \cos\alpha = \{\alpha = \phi - \theta\} = |z||w| \cdot \cos(\phi - \theta)$

Å andra sidan

$\bar{z} w = r\rho[\cos(\phi-\theta) + i\sin(\phi-\theta)]$

$\therefore \langle z, w\rangle = \operatorname{Re} w\bar{z} = \operatorname{Re} z\bar{w}$

Följd:

$|\langle z, w\rangle| = |\operatorname{Re} z\bar{w}| \leq |z\bar{w}| = |z||w|$

Linje

ekv. $ax + by = c$, $(*)$ $a, b, c \in \mathbb{R}$

Sätt $z = x + iy$, $A = a + ib$

$z\bar{A} = ax + yb + i(\ldots)$

$(*)$: $\operatorname{Re} z\bar{A} = c$, ekv. för rät linje.

Viktig formel

$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = |z|^2 + |w|^2 + z\bar{w} + w\bar{z} =$

$= |z|^2 + |w|^2 + z\bar{w} + \overline{\bar{w}z} = |z|^2 + |w|^2 + 2\operatorname{Re} z\bar{w}$

(ty $A + \bar{A} = 2\operatorname{Re} A$)

ex $|z-2|^2 = |z+2|^2 \Rightarrow \cancel{|z|^2} + \cancel{4} - 4\operatorname{Re} z = \cancel{|z|^2} + \cancel{4} + 4\operatorname{Re} z$

$\Rightarrow \operatorname{Re} z = 0$

Cirkel: $|z - c| = r$

Övning 1.2.5

$|z+2|+|z-2|=5$

$|z+2|^2=(5-|z-2|)^2$

$|z+2|^2=25+|z-2|^2-10|z-2|$

$(x+2)^2+\cancel{y^2}=25+(x-2)^2+\cancel{y^2}-10|z-2|$

$4x=25-4x-10|z-2|$

$10|z-2|=25-8x$

$100((x-2)^2+y^2)=625+64x^2-400x$

$100x^2+100y^2-\cancel{400x}+400=625+64x^2-\cancel{400x}$

$36x^2+100y^2=225$

har formen $ax^2+by^2=c \Rightarrow$ ellips.

Extra övning

$z_1, z_2, z_3 \in \mathbb{C}$

$z_1^2+z_2^2+z_3^2=z_1z_2+z_1z_3+z_2z_3$

$\Longleftrightarrow$

z_1, z_2, z_3 bildar hörn i en <u>liksidig</u> triangel.