Föreläsning 6/9-13

Elementära funktioner

(i) $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$, $z = x+iy$

(ii) $e^z = w \iff z = \log w$, $w \neq 0$

$\log w = \log|w| + i\arg(w)$

(ex. $\log 1 = i2k\pi$ ty $e^{i2k\pi} = 1$)

(iii) $a \in \mathbb{C}$, $a \neq 0$

$a^z = e^{z\log a}$

(iv) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Motivering:

$z = y$ reellt

$$\frac{e^{iy} + e^{-iy}}{2} = \frac{\cos y + i\sin y}{2} + \frac{\cos y - i\sin y}{2} = \cos y$$

$\therefore$ Formeln stämmer om z reellt.

Anm: $e^z \cdot e^w = e^{z+w}$

Koll: $z = x+iy$, $w = u+iv$

$$e^z \cdot e^w = e^x(\cos y + i\sin y)e^u(\cos v + i\sin v) =$$
$$= e^{x+u}[\cos y\cos v - \sin y\sin v + i(\cos y\sin v +$$
$$+ \sin y\cos v)] =$$
$$= e^{x+u}(\cos(y+v) + i\sin(y+v)) =$$
$$= e^{z+w}$$

Spec: $e^z \cdot e^{-z} = e^0 = 1$

$\therefore$ $e^z \neq 0$ och $\log 0$ finns ej.

$e^{\bar{z}} = e^x(\cos y - i\sin y) = \overline{e^z}$

spec: $e^{iy}\overline{e^{iy}} = e^{iy}e^{\overline{iy}} = e^{iy}e^{-iy} = 1$

$\therefore\ |e^{iy}| = 1$

dvs $\cos^2 y + \sin^2 y = 1$

Kurvintegraler

Vad är en kurva i $\mathbb{C}$?

Definition

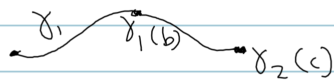
En glatt kurva γ är en glatt funktion

$\gamma(t) = x(t) + iy(t) : [a,b] \longrightarrow \mathbb{C}$

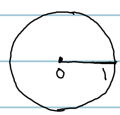
En styckvis glatt kurva är t.ex

$\gamma_1 : [a,b] \longrightarrow \mathbb{C}$ och $\gamma_2 : [b,c]$ glatta

$\gamma_1(b) = \gamma_2(b)$



ex



$\gamma(t) = \cos t + i\sin t = e^{it}$

$0 \le t \le 2\pi$

ex

$$\gamma(t) = \begin{cases} e^{it} & 0 \le t \le \pi \\ t - \pi - 1 & \pi \le t \le \pi + 2 \end{cases}$$

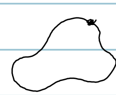
Definition

γ är sluten om $\gamma(a) = \gamma(b)$

Definition

γ är enkel om $t_1 \neq t_2 \Longrightarrow \gamma(t_1) \neq \gamma(t_2)$

förutom $t_1 = a$, $t_1 = b$



Jordans kurvsats

En enkel sluten kurva delar planet i två delar.

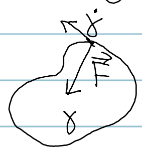
Reell kurvintegral

P och Q : $\mathbb{C} \to \mathbb{R}$

Definition

$$\int_\gamma P\,dx + Q\,dy = \int_a^b \left[P(\gamma(t))\dot{x} + Q(\gamma(t))\dot{y}\right] dt$$

$\dot{\gamma} = (\dot{x}, \dot{y})$ är tangentvektor till γ



Skriv $\vec{F} = (P, Q)$ vektorfält

$P\dot{x} + Q\dot{y} = \langle \vec{F}, \dot{\gamma} \rangle$ skalärprodukt

$= |\vec{F}||\dot{\gamma}|\cos\alpha$ kraftens komp. i $\dot{\gamma}$:s riktning

$\therefore \int_\gamma P\,dx + Q\,dy =$ arbetet som $\vec{F}$ utför

Ex

e^{it}

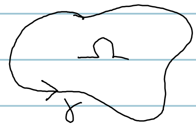
$\vec{F} = -(x, y)$

$$-\int_\gamma x\,dx + y\,dy = 0$$

$$\langle \vec{F}, \dot{\gamma} \rangle = 0$$

Greens formel

Anta γ enkel och sluten och begränsar området Ω.



$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

(ex)

$P = -x, \quad Q = -y, \quad \gamma = e^{it}$

$$\int_\gamma -x\,dx - y\,dy = -\iint_{|z|<1} \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\,dx\,dy = 0$$

Anm:

P och Q kan vara komplexvärda.
(och Greens formel gäller fortfarande)

Definition

$g(t) = \delta(t) + i\tau(t)$

$$\int_a^b g(t)\,dt = \int_a^b \delta(t)\,dt + i\int_a^b \tau(t)\,dt$$

Triangelolikheten (för integraler)

$$\left|\int_a^b g(t)\,dt\right| \leq \int_a^b |g(t)|\,dt$$

bevis senare.

Komplex kurvintegral

γ (styckvis) glatt

f komplexvärd

Definition

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\dot{\gamma}(t)\,dt$$

Obs $f = u + iv$, $\gamma = x + iy$

$$\int_\gamma f\,dz = \int_a^b (u+iv)(\dot{x}+i\dot{y})\,dt = \int_a u\dot{x} - v\dot{y} + i(v\dot{x} + u\dot{y})\,dt$$

$$= \int_\gamma u\,dx - v\,dy + i(v\,dx + u\,dy)$$

ex

$f(z) = z^m$, $m \in \mathbb{Z}$

$$\int_{|z|=1} z^m\,dz = \left\{ \begin{matrix} \gamma(t) = e^{it} \\ 0 \le t \le 2\pi \end{matrix} \right\} = \int_0^{2\pi} (e^{it})^m \dot{\gamma}\,dt = \star$$

$$\left(\text{Obs. } \frac{d}{dt} e^{it} = \frac{d}{dt}(\cos t + i\sin t) = = -\sin t + i\cos t = ie^{it} \right)$$

$$\star = i \int_0^{2\pi} e^{i(m+1)t}\,dt = \frac{1}{m+1} \int_0^{2\pi} \frac{d}{dt} e^{i(m+1)t}\,dt =$$

$$= \frac{1}{m+1} \int_0^{2\pi} \frac{d}{dt} e^{i(m+1)t}\,dt =$$

$$= \frac{1}{m+1} \left[e^{i(m+1)t} \right]_0^{2\pi} = 0 \qquad \text{OK om } m+1 \neq 0$$

forts. →

om $m+1=0$ dvs $m=-1$

$$i\int_0^{2\pi} e^{i(m+1)t}\,dt = i\int_0^{2\pi} 1\,dt = 2\pi i$$

$$\int_{|z|=1} z^m\,dz = \begin{cases} 2\pi i, & m=-1 \\ 0 & \text{annars} \end{cases}$$

Greens formel (komplex form)

γ enkel, sluten och begränsar Ω

$$\int_\gamma f(z)\,dz = i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dx\,dy$$

Green gäller om f glatt i Ω

ex

$f(z) = z = x+iy$

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 1 + i^2 = 0$$

$$\int_{|z|=1} z\,dz = 0$$

ex

$f(z) = 1/z = 1/(x+iy)$

$$\frac{\partial f}{\partial x} = -\frac{1}{(x+iy)^2} \qquad \frac{\partial f}{\partial y} = \frac{-i}{(x+iy)^2}$$

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

~~$\int_{|z|=1} 1/z\,dz = 0$~~ $\quad z \neq 0$

Bevis av komplexa Green

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\dot\gamma(t)\,dt = \int_\gamma f(\gamma(t))(\dot x + i\dot y)\,dt =$$

$$= \int_\gamma f\,dx + i\int_\gamma f\,dy = \iint_\Omega -\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x}\,dxdy =$$

$$= i\iint_\Omega \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\,dxdy \qquad \blacksquare$$

Definition

$|\gamma|$ = längden av $\gamma = \int_a^b \sqrt{\dot x^2 + \dot y^2}\,dt = \int_a^b |\dot\gamma|\,dt$

Proposition

Antag $|f| \leq M$ på γ

Då $\left|\int_\gamma f\,dz\right| \leq M|\gamma|$

Bevis

$$\left|\int_\gamma f\,dz\right| = \left|\int_a^b f(\gamma(t))\dot\gamma\,dt\right| \overset{\Delta\text{-olikheten}}{\leq} \int_a^b |f(\gamma(t))||\dot\gamma|\,dt$$

$$\leq M\int_a^b |\dot\gamma|\,dt = M|\gamma| \qquad \blacksquare$$