Föreläsning 24/9-13

Residyen (forts/rep.)

Låt f vara holomorf i $\{z;\ 0<|z-z_0|<R\}$

Då är $\frac{1}{2\pi i}\int_{|z-z_0|=s} f(z)\,dz$, $0<s<R$ ober. av s.

Definition

$$\mathrm{Res}(f,z_0)=\frac{1}{2\pi i}\int_{|z-z_0|=s} f(z)\,dz$$

Hur beräknar man Res?

1) Säg $f(z)=\frac{F(z)}{G(z)}$ F, G holo i $\{|z-z_0|<R\}$

där $G(z_0)=0$ till ordning $m\geq 1$, $F(z_0)\neq 0$.

Då $\exists$ g holo s.a $G(z)=(z-z_0)^m g(z)$, $g(z_0)\neq 0$

Då $f=\frac{F}{G}=\frac{F/g}{(z-z_0)^m}=\frac{H(z)}{(z-z_0)^m}$ där H holo. $=F/g$

$$\mathrm{Res}(f,z_0)=\frac{H^{(m-1)}(z_0)}{(m-1)!}$$

Kan vara jobbigt att räkna ut $H^{(m-1)}(z_0)$.

ex) $f=\frac{\cos z}{(\sin z)^m}$, $z_0=0$

enkelt när $m=1$

Sats

Om $m=1$ och $f=F/G$ så $\mathrm{Res}(f,z_0)=\frac{F(z_0)}{G'(z_0)}$

Bevis

$$f = \frac{F}{G} = \frac{H}{(z-z_0)}, \quad \text{Vet att } \operatorname{Res}(f, z_0) = \frac{H^{(m-1)}(z_0)}{(m-1)!} =$$

$$= H(z_0) = \lim_{z \to z_0} f \cdot (z-z_0) = \lim_{z \to z_0} \frac{F}{G/(z-z_0)} =$$

$$= \lim_{z \to z_0} \frac{F}{\left(\frac{G(z)-G(z_0)}{z-z_0}\right)} = \frac{F(z_0)}{G'(z_0)} \quad \blacksquare$$

ex $f(z) = \dfrac{\cos z}{\sin z}$, $z_0 = 0$

$$\operatorname{Res}(f, z_0) = \frac{\cos 0}{\cos 0} = 1$$

ex $\operatorname{Res}\left(\dfrac{e^z}{z^2+1}, i\right) = \dfrac{e^i}{2i}$

2) Hur hanteras t.ex $f(z) = e^{1/z}$?

$$e^w = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \dots, \quad e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \dots$$

$$\operatorname{Res}(f, 0) = \frac{1}{2\pi i} \int_{|z|=s} f(z)\,dz = \sum \frac{1}{n!} \frac{1}{2\pi i} \int_{|z|=s} z^{-n} =$$

$$= \left\{ \int_{|z|=s} \frac{dz}{z^n} = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases} \right\} = 1$$

Allmänt: f holo i $0 < |z - z_0| < r$?

Laurentserier

Sats

Låt f holo i $\{z;\ r \le |z-z_0| \le R\}$

Då kan f skrivas:

$$f(z) = \sum_0^\infty a_k (z-z_0)^k + \sum_1^\infty b_k (z-z_0)^{-k} = f_1 + f_2$$

f_1 holo i $|z-z_0| \le R$

f_2 holo i $r \le |z-z_0|$ (f_2 kallas principaldelen)

$\text{Res}(f, z_0) = b_1$

Bevis

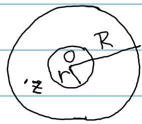
1) $$f(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w-z}\,dw, \quad z_0 = 0$$

Motivering:

Låt $\Gamma = \{|w| = R\}$

$\gamma = \{|w| = r\}$

$\gamma_\delta = \{|w-z| = \delta\}$ $\quad \delta << 1$



Då $$\int_\Gamma \frac{f(w)}{w-z}\,dw - \int_\gamma \frac{f(w)}{w-z}\,dw - \int_{\gamma_\delta} \frac{f(w)}{w-z}\,dw = 0$$

$$\left(= i \iint_\Omega \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{f(w)}{w-z}\,dx\,dy = 0 \text{ enl. Green}\right)$$

Följd: $\int_\Gamma - \int_\gamma = \int_{\gamma_\delta} \xrightarrow[\delta \to 0]{} f(z)$ som i beviset för Cauchys formel.

$\because f = f_1 + f_2$

$$f_1(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w-z}\,dw = \sum_0^\infty a_k z^k \text{ (som förut)}$$

$$f_2(z) = -\frac{1}{2\pi i}\int_{|w|=r} \frac{f(w}{w-z}\,dw = ?$$

Hur utvecklade vi f_1?

Jo: $f_1(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w-z}\,dw$

vet att $r<|z|<R$

$$f_1(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{1-\frac{z}{w}}\,\frac{dw}{w} = \frac{1}{2\pi i}\int_{|w|=R} f(w)\sum_0^\infty \left(\frac{z}{w}\right)^k \frac{dw}{w}$$

$$= \sum_0^\infty z^k \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w^{k+1}}\,dw$$

Nu: $f_2(z) = \frac{-1}{2\pi i}\int_{|w|=r} \frac{f(w)}{w-z}\,dw =$

$$= -\frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{\frac{w}{z}-1}\,\frac{dw}{z} = \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)}{1-\frac{w}{z}}\,\frac{dw}{z} =$$

$$= \left\{\left|\frac{w}{z}\right|<1\right\} = \frac{1}{2\pi i}\int_{|w|=r} f(w)\sum_0^\infty \left(\frac{w}{z}\right)^k \frac{dw}{z} =$$

$$= \sum_0^\infty \frac{1}{z^{k+1}}\frac{1}{2\pi i}\int_{|w|=r} f(w)\,w^k\,dw = (k+1\to k) =$$

$$= \sum_1^\infty z^{-k}\frac{1}{2\pi i}\int_{|w|=r} f(w)\,w^{k-1}\,dw = \sum_1 b_k z^{-k}$$

$$b_1 = \frac{1}{2\pi i}\int_{|w|=r} f(w)\,dw = \mathrm{Res}(f,0).$$

$\blacksquare$

I praktiken?

ex $\frac{z^2}{(z-1)^2} = f \qquad z_0 = 1$

Taylorutv. z^2 runt 1.

forts. →

$z^2 = (z-1+1)^2 = (z-1)^2 + 2(z-1) + 1$

alt. $z^2 = g(z)$

$\Rightarrow g(z) = g(1) + \frac{g'(1)}{1!}(z-1) + \frac{g''(1)}{2!}(z-1)^2 + \ldots =$

$= 1 + 2(z-1) + (z-1)^2 + \ldots$

$\therefore\ f(z) = \frac{1 + 2(z-1) + (z-1)^2}{(z-1)^2} = \frac{1}{(z-1)^2} + \frac{2}{(z-1)} + 1$

$\text{Res}(f, z_0) = b_1 = 2$

ex $\frac{\sin z}{z^3} = f(z)$. Taylorutv. $\sin z$.

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$

$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \ldots$

$\text{Res}(f, 0) = 0$

ex $f(z) = \cot z = \frac{\cos z}{\sin z}$, $z_0 = 0$

$\left(\text{Ansätt: } f(z) = \sum_0^\infty a_k z^k + \sum_1^\infty b_k z^{-k} \right)$

(i) enkelpol, dvs. nämnaren $\sin z$ har nollst. av ordning $= 1 \Rightarrow \sum_1^\infty b_k z^{-k} = \frac{b_1}{z}$

ty $\sin z = z\,g(z)$, $g(0) \neq 0$

$f(z) = \frac{\cos z}{\sin z} = \frac{\cos z / g(z)}{z} = \frac{c_0 + c_1 z + c_2 z^2 + \ldots}{z} =$

$= \frac{c_0}{z} + c_1 + c_2 z + \ldots$ Dessutom f udda, $f(-z) = -f$

$\Rightarrow f(z) = \frac{\cos z}{\sin z} = \frac{b_1}{z} + a_1 z + a_2 z^3 + \ldots$

$\therefore\ \cos z = \sin z \left(\frac{b_1}{z} + a_1 z + a_3 z^3 + \ldots \right)$

$$\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)=\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\right)\left(\frac{b_1}{z}+a_1 z+\dots\right)$$

$z^0: \; b_1 = 1$

$z^2: \; a_1 - \frac{b_1}{3!} = -\frac{1}{2!} \implies a_1 = -1/3$

$\therefore \; \mathrm{Res}(f,0) = 1 \quad (= b_1)$

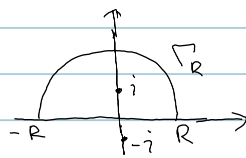
Residyen (satsen)

D enkelt sammanh., f holo i D, förutom i punkterna $z_1, \dots, z_n$. γ enkel sluten kurva i D.

Då
$$\int_\gamma f\,dz = 2\pi i \sum_{z_j \text{ innanför } \gamma} \mathrm{Res}(f, z_j)$$

ex
$$\int_{|z|=5} \frac{e^z}{(z-2)(z-3)}\,dz = 2\pi i\left(\mathrm{Res}_2 + \mathrm{Res}_3\right) =$$
$$= 2\pi i\left(\frac{e^2}{2-3}+\frac{e^3}{3-2}\right)$$

ex
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$



$$\int_{\Gamma_R} = 2\pi i \,\mathrm{Res}\left(\frac{1}{1+z^2}, i\right) = \frac{2\pi i}{2i} = \pi$$