Föreläsning 9/10-13

Fouriertransformen

$$\mathcal{F}(u)(x) = \hat{u}(x) = \int_{-\infty}^{\infty} u(t) e^{-itx} \, dt$$

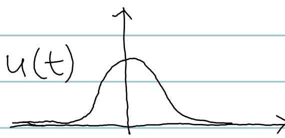
Egenskaper:

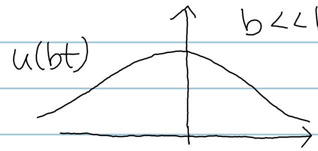
(i) $\mathcal{F}(u')(x) = ix\hat{u}(x)$

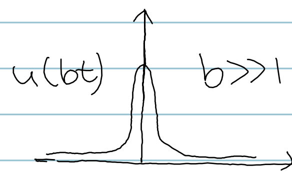
(ii) $\widehat{u(bt)} = \frac{1}{|b|} \hat{u}\left(\frac{x}{b}\right)$

ex $u(t) = e^{-t^2}$

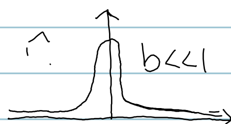
$u(bt) = e^{-(bt)^2}$

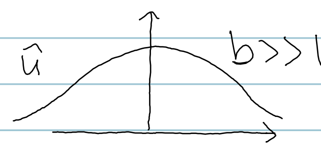






$\hat{u} \simeq e^{-x^2/4b^2} \cdot \frac{1}{|b|}$





ex **Värmeledning**

$u(t,s)$ = temp. i punkt s vid tid t.

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2}$, $u(0,s) = f(s)$

Sätt $\phi(t,x) = \int_{-\infty}^{\infty} u(t,s) e^{-isx} \, ds$

forts. →

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2} \Longrightarrow \frac{\partial \phi}{\partial t} = -x^2 \phi ,$$

ty $\frac{\partial \phi}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-isx} ds = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial s^2} e^{-isx} ds =$

$= (ix)^2 \int_{-\infty}^{\infty} u e^{-isx} ds = -x^2 \phi$

$\therefore \phi(t,x) = e^{-tx^2} \cdot c(x)$

$\phi(0,x) = \hat{f}(x) = c(x)$

$\therefore \phi(t,x) = \hat{f}(x) e^{-tx^2}$

Säg att $K_t(s)$ {funktion av s, t konstant}

uppfyller $\hat{K}_t(x) = e^{-tx^2}$

Då $\phi(t,x) = \hat{f}(x) \hat{K}_t(x) = \widehat{f * K_t}$

$\therefore u(t,s) = f * K_t$ där $\hat{K}_t = e^{-tx^2}$

Vet att $u(s) = e^{-s^2/2}$

$\Longrightarrow \hat{u}(x) = \sqrt{2\pi} e^{-x^2/2}$

Skalning $\Longrightarrow \widehat{u(bs)} = \sqrt{2\pi} e^{-x^2/2b^2} \frac{1}{b}$, $b > a$

Välj b; $\frac{1}{2b^2} = t$

Då $\widehat{u(bs)} = \sqrt{2\pi} e^{-tx^2} \frac{1}{b}$, $b = 1/\sqrt{2t}$

$u(bs) = e^{-s^2/4t}$

$e^{-tx^2} = \widehat{\frac{b}{\sqrt{2\pi}} u(bs)} = \hat{K}_t$

$\therefore K_t = \frac{b}{\sqrt{2\pi}} u(bs) = e^{-s^2/4t} \frac{1}{\sqrt{2\pi}\sqrt{2t}}$

$\Longrightarrow u(t,s) = f * K_t$

ex $u(t) = e^{-t^2/2}$

$\hat{u}(x) = \sqrt{2\pi}\, e^{-x^2/2}$

Fouriertransf. av en Gauss-funktion blir en ny Gauss-funktion.

u löser $u' = -tu$

$ix\hat{u} = -i \frac{d}{dx}\hat{u}$

$\frac{d\hat{u}}{dx} = -x\hat{u}$

$\therefore \hat{u}(x) = C e^{-x^2/2}$

Inversionsformeln

Diracs deltafunktion

$$\delta_a(t) = \begin{cases} 0 & t < a \\ \text{odef.} & \\ 0 & t > 0 \end{cases}$$

och $\int_{-\infty}^{\infty} \delta_a(t)\,dt = 1$, t.o.m $\int_{-\infty}^{\infty} \delta_a(t) w(t)\,dt = w(a)$

$$\hat{\delta}_a(x) = \int \delta_a(t) \underbrace{e^{-itx}}_{w(t)}\,dt = e^{-iax}$$

① $u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(x) e^{itx}\,dx$

② (Plancherel)

$$\int_{-\infty}^{\infty} |u(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}|^2\,dx$$

③ Allm.

$$\int_{-\infty}^{\infty} u\bar{v}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}\bar{\hat{v}}\,dx$$

Formellt: ③ $\Rightarrow$ ①

Tag $v = \delta_a$

$$\int u \delta_a = \frac{1}{2\pi} \int \hat{u} \overline{\hat{\delta}_a} \, dx$$

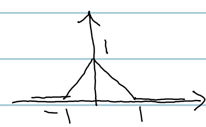
$$u(a) = \frac{1}{2\pi} \int \hat{u}(x) e^{iax} \, dx$$

ex Räkna ut

$$\int_{-\infty}^{\infty} \left(\frac{1-\cos x}{x^2} \right)^2 dx = \int |u|^2 dx \quad \text{om } u = \frac{1-\cos x}{x^2}$$

$$= 2\pi \int |v(t)|^2 dt \quad \text{om } u = \hat{v}$$

$$u = \frac{1-\cos x}{x^2} = \hat{v} \frac{1}{2} \quad \text{om} \quad v =$$



∵ Plancherel $\Rightarrow \int |u|^2 dx = \frac{1}{4} \int |\hat{v}|^2 = \frac{2\pi}{4} \int |v|^2 =$

$$= \frac{4\pi}{4} \int_0^1 v^2 dt = \pi \int_0^1 (1-t)^2 dt = \pi/3$$

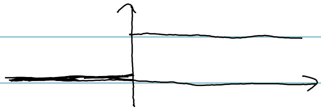
Definition (Laplacetransform)

Säg $u(t) = 0$ om $t < 0$ och $|u(t)| \le Me^{at}$ om $t > 0$

Då $\mathcal{L}(u)(s) = \tilde{u}(s) = \int_0^\infty u(t) e^{-st} dt$

Definierad då $\mathrm{Re}(s) > a$ ($\Rightarrow$ integralen ändlig).

ex $u = 1$, egentligen $u = H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$



(brukar kallas för Heaviside)

$$\hat{u}(s) = \int_0^\infty 1 \cdot e^{-st}\,dt = \frac{1}{s}$$

ex $\delta_a = u$

$$\tilde{\delta}_a(s) = \int_0^\infty \delta_a(t)\, e^{-st}\,dt = e^{-as}$$

ex $\mathcal{L}(t^k)(s) = \int_0^\infty t^k e^{-ts}\,dt = \{\text{partiell integr.}\} =$

$$= \left[\frac{e^{-st}}{-s}\, t^k\right]_0^\infty + \frac{k}{s}\int_0^\infty t^{k-1} e^{-ts}\,dt = \dots = \frac{k!}{s^k}$$

ex $u(t) = e^{at}$

$\mathcal{L}(e^{at}) = \int_0^\infty e^{at-st}\,dt = \frac{1}{s-a}$ OK, även om $a \in \mathbb{C}$

ex $\mathcal{L}(\sin t) = \frac{\mathcal{L}(e^{it}) - \mathcal{L}(e^{-it})}{2i} = \frac{1}{2i}\left[\frac{1}{s-i} - \frac{1}{s+i}\right] = \frac{1}{s^2+1}$

ex $\mathcal{L}(\cos t)(s) = \frac{s}{s^2+1}$

Formel: $\mathcal{L}(u')(s) = \int_0^\infty u'(t)\, e^{-st}\,dt = \left[u(t)e^{-st}\right]_0^\infty + s\int_0^\infty u e^{-st}\,d$

$$= -u(0) + s\tilde{u}(s)$$

Allmännare:

$$\mathcal{L}(u^{(k)})(s) = s^k \hat{u}(s) - (s^{k-1}u(0) + s^{k-2}u'(0) + \ldots + u^{(k-1)}(0))$$

Storgruppsövning 9/10-13

3.3 Möbiusavbildningar

$\left(\begin{array}{l} D_T = \mathbb{C} \setminus \{-\frac{d}{c}\} \\ V_T = \mathbb{C} \setminus \{\frac{a}{c}\} \end{array} \right)$

Definition

En rationell fkn av formen

$$T(z) = \frac{az+b}{cz+d} \quad \text{där } a,b,c,d \in \mathbb{C} \text{ och } ad-bc \neq 0$$

kallas för en Möbiusavbildning (M.a).

Några viktiga egenskaper:

(i) $ad-bc \neq 0 \Rightarrow T$ injektiv $\Rightarrow T^{-1}$ existerar.

Kan visa att $T^{-1}(w) = \frac{-dw+b}{cw-a}$

(ii) Låt $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ Riemannsfären

Varje M.a är en bijektiv avb. från $\hat{\mathbb{C}}$ till $\hat{\mathbb{C}}$

(iii) Om S och T är M.a, så är $S \circ T$ M.a

(iv) Låt "cirkel" = cirkel eller {rät linje} $\cup \{\infty\}$

(dvs "cirkel" = cirkel på $\hat{\mathbb{C}}$).

Om T M.a så avbildar T "cirklar" på "cirklar"