Föreläsning 15/10-13

Z-transform.

Låt a_j ; $j = 0, 1, 2, \ldots$ vara en talföljd,
$|a_j| \leq M r_0^j$

Skriv $a = \{a_j\}$

Definition

$Z(a)(z) = Z(\{a_j\})(z) = \sum_0^\infty a_j / z^j$

Konvergerar om $|z| > r_0$;

$$\left|\frac{a_j}{z^j}\right| \leq \frac{M r_0^j}{|z|^j} = M\left(\frac{r_0}{|z|}\right)^j$$

$$\therefore \sum \left|\frac{a_j}{z^j}\right| < \infty$$

ex $a_j = 1$; $a = \{1\}$, $r_0 = 1$

$$Z(a)(z) = \sum_0^\infty \frac{1}{z^j} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1} \qquad |z| > 1$$

ex $$\frac{d}{dz} \frac{z}{z-1} = \frac{d}{dz}\left(1 + \frac{1}{z-1}\right) = -\frac{1}{(z-1)^2}$$

$$\frac{d}{dz} \sum_0^\infty \frac{1}{z^j} = -\sum_0^\infty \frac{j}{z^{j+1}}$$

$$\therefore \sum_0^\infty \frac{j}{z^{j+1}} = \frac{1}{(z-1)^2} \implies \sum_0^\infty \frac{j}{z^j} = \frac{z}{(z-1)^2}$$

$\therefore Z(\{j\}) = z/(z-1)^2$

Entydighet: $Z(a)$ bestämmer a.

Z-transf. är en diskret variant av Laplacetransf.:

Låt $u(t)$ def. för $t > 0$.

$|u(t)| \le M e^{ct}$, $\mathcal{L}(u) = \int_0^\infty u(t) e^{-st} dt \approx \sum_0^\infty u(j) e^{-sj} =$

$= \sum_0^\infty \frac{u(j)}{z^j}$, om $z = e^{+s}$

Därför (?) har Z-transf. egenskaper som liknar Laplacetransf.:

(i) $Z(\lambda a + \mu b) = \lambda Z(a) + \mu Z(b)$, $\lambda, \mu \in \mathbb{C}$

(ii) $Z(a * b) = Z(a) Z(b)$

Här är $a * b = c = \{c_n\}$

$c_n = \sum_{k=0}^n a_{n-k} b_k = \sum_{k=0}^n b_{n-k} a_k = \sum_{\substack{k+j=n \\ k,j>0}} a_k b_j$

(Jfr. $u * v(t) = \int_0^t u(t-s) v(s) ds$)

$a_n b_0 + a_{n-1} b_1 + \dots + a_0 b_n$

Vi def. också $S(a) = (a_1, a_2 \dots)$, dvs

$(S(a))_j = a_{j+1} \quad j = 0, 1, 2, \dots$

$S^N(a)_j = a_{j+N}$

(iii) $Z(S(a)) = z[Z(a) - a_0]$

(iv) $Z(S^N(a)) = z^n \left[Z(a) - a_0 - \frac{a_1}{z} - \dots - \frac{a_{N-1}}{z^{N-1}}\right]$

Bevis av (iv):

$z^N \left[Z(a) - (a_0 + \frac{a_1}{z} + \dots + \frac{a_{N-1}}{z^{N-1}})\right] = z^N \sum_N^\infty \frac{a_j}{z^j} =$

$= z^N \sum_0^\infty \frac{a_{k+N}}{z^{k+N}} = \sum_0^\infty \frac{a_{k+N}}{z^k} = Z(S^N(a))$

$\blacksquare$

Bevis av (ii)

$Z(a) = \sum_0^\infty \frac{a_j}{z^j}$, $Z(b) = \sum_0^\infty \frac{b_j}{z^j}$

$Z(a)Z(b) = \sum_{j,k} \frac{a_j b_k}{z^j z^k} = \{j+k=n\} = \sum_n \sum_{j+k=n} \frac{a_j b_k}{z^n} =$

$= \sum_0^\infty \frac{c_n}{z^n}$; $c_n = \sum_{j+k=n} a_j b_k$ ▨

ex $a_{j+1} = c \cdot a_j$ $(*)$, $c \in \mathbb{C}$

$a_0, c a_0, c^2 a_0, \dots$

z-transf.:

$(*) \Rightarrow S(a) = c \cdot a$

$z[Z(a) - a_0] = cZ(a)$, $(z-c)Z(a) = z a_0$

$Z(a) = \frac{z}{z-c} \cdot a_0$

Men $\frac{z}{z-c} = \frac{1}{1-\frac{c}{z}} = \sum_0^\infty \frac{c^j}{z^j} = Z(\{c^j\})$.

$\therefore$ $a_j = a_0 c^j$.

ex $b_0 = a_0$ och $b_{j+1} = a_{j+1} - a_j$, $\{a_j\}$ given.

Lös ut a_j i termer av b_j

$b_0 = a_0$

$b_1 = a_1 - a_0$

$b_2 = a_2 - a_1$ $\Longrightarrow$ $b_0 + b_1 + \dots + b_j = a_j$

$\vdots$

$b_j = a_j - a_{j-1}$

$\vdots$

Med z-transf.:

$$S(\{b_j\}) = S(\{a_j\}) - \{a_j\}, \quad b_0 = a_0$$

$$z[Z(b) - b_0] = z[Z(a) - a_0] - Z(a)$$

$$(z-1)Z(a) = zZ(b)$$

$$Z(a) = \frac{z}{z-1} Z(b) = Z(\{1\})Z(b)$$

$$\therefore \quad a = \{1\} * b, \quad a_n = \sum_{j=0}^{n} 1_{n-j} b_j = \sum_{0}^{n} b_j$$

Fibonaccital

$$a_0 = 0, \ a_1 = 1, \ \ldots, \ a_{j+2} = a_{j+1} + a_j$$

$$\Longrightarrow 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

kvoten: $\frac{8}{5}, \frac{13}{8}, \frac{21}{13} \longrightarrow \frac{1+\sqrt{5}}{2}$ (gyllene snittet)

Talen uppfyller

$$S^2 a = S(a) + a$$

$$z^2[Z(a) - \underbrace{a_0}_{=0} - \tfrac{a_1}{z}] = z[Z(a) - \underbrace{a_0}_{=0}] + Z(a)$$

$$(z^2 - z - 1)Z(a) = z^2 \frac{a_1}{z} \pm z$$

$$\Longrightarrow Z(a) = \frac{z}{z^2 - z - 1}$$

$$z^2 - z - 1 = 0 \Longrightarrow z = \frac{1 \pm \sqrt{5}}{2}$$

Partialbråksuppdela:

$$\frac{z}{z^2 - z - 1} = \frac{A}{z - z_0} + \frac{B}{z - z_1} \quad \text{där } z_0 = \frac{1+\sqrt{5}}{2}, \ z_1 = \frac{1-\sqrt{5}}{2}$$

$$\Longrightarrow A = z_0/(z_0 - z_1), \quad B = z_1/(z_1 - z_0)$$

Sen $\frac{1}{z - z_0} = \frac{1}{z} \frac{1}{1 - \frac{z_0}{z}} = \frac{1}{z} \sum_{0}^{\infty} \left(\frac{z_0}{z}\right)^j = \sum_{0}^{\infty} \frac{z_0^{\,j}}{z^{j+1}}$

och $\frac{1}{z-z_1} = \sum_0^\infty \frac{z_1^j}{z^{j+1}}$

$\frac{A}{z-z_0}+\frac{B}{z-z_1} = A\sum_0^\infty \frac{z_0^j}{z^{j+1}} + B\sum_0^\infty \frac{z_1^j}{z^{j+1}} =$

$= \frac{1}{z_0-z_1}\sum_0^\infty \frac{z_0^{j+1}}{z^{j+1}} + \frac{1}{z_1-z_0}\sum_0^\infty \frac{z_1^{j+1}}{z^{j+1}} = \left\{z_0-z_1=\sqrt{5}\right\} =$

$= \frac{1}{\sqrt{5}}\left[\sum_0^\infty \frac{z_0^k}{z^k} - \sum_0^\infty \frac{z_1^k}{z^k}\right]$ (kan gå från $k=0$, ty det gör ingen skillnad då de tar ut varandra)

$\therefore$ $Z(a) = \frac{z}{z^2-z-1} = \frac{1}{\sqrt{5}}\sum_0^\infty \frac{z_0^j - z_1^j}{z^j}$

$\therefore$ $a_j = \frac{1}{\sqrt{5}}\left[z_0^j - z_1^j\right] =$ (ty $Z(a) = \sum_0^\infty \frac{a_j}{z^j}$)

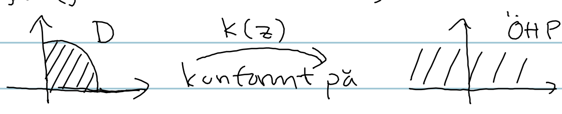
$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^j - \left(\frac{1-\sqrt{5}}{2}\right)^j\right]$

$\frac{a_{j+1}}{a_j} = \frac{z_0^{j+1} - z_1^{j+1}}{z_0^j - z_1^j} = \frac{z_0 - z_1\left(\frac{z_1}{z_0}\right)^j}{1-\left(\frac{z_1}{z_0}\right)^j} \longrightarrow z_0 = \frac{1+\sqrt{5}}{2}$

ty $\left|\frac{z_1}{z_0}\right| < 1$

Tentauppg. (januari 2013/3)

Avbilda



Dela upp i steg:

(1) Avbilda D på D_1:

$|z|<1$, $\operatorname{Im} z > 0$

$z \to z^2$

(2) Avbilda D_1 på 1:sta kvadranten:

avbilda cirkeln $\to$ Im-axeln och Re-axeln $\to$ Re-ax.

Välj T möbius s.a

$T(-1) = 0$, $T(1) = \infty$

Då duger T (varför?)

Tag $T(z) = \frac{z+1}{z-1}$ forts. följer!