Forts från Förel. Mån LV4 !

Mån LV4

Def. Ett reguljärt Sturm-Liouville problem i ett begränsat intervall $[a,b]$ ges av en formellt självadj. operator $Lf = (rf')' + pf$ där r, r', p är reellvärda och kontinuerliga och $r > 0$ i $[a,b]$.

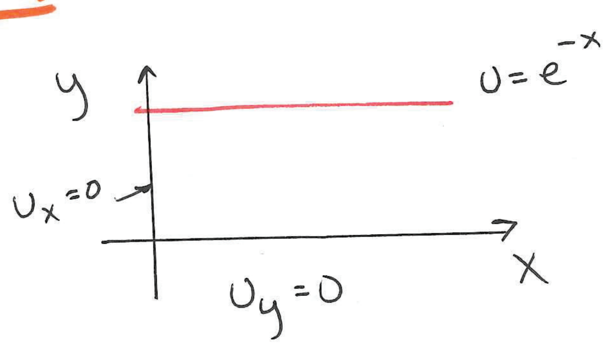
Kan även ges av:

- självadj. randvillkor $B_1(f) = 0$, $B_2(f) = 0$
- en viktsfunktion w som är kontinuerlig och positiv i $[a,b]$

$\lambda \in \mathbb{C}$ kallas egenvärde till problemet om ekvationen $Lf + \lambda w f = 0$ har en lösning $f \in C^{(2)}$, $f \not\equiv 0$ med $B_1(f) = B_2(f) = 0$

f kallas egenfunktion

7.4.6 $\Delta u = 0$, $x > 0$, $0 < y < 1$



Ta cos-transformen i x-variabeln.

sätt $w(\xi, y) = \mathcal{F}_c u(\xi, y) = \int_0^\infty u(x,y) \cos x\xi \, dx$

Då blir $u(x,y) = \frac{2}{\pi}\int_0^\infty w(\xi,y)\cos x\xi \, d\xi$

och $u_x(0,y) = -\frac{2}{\pi}\int_0^\infty w(\xi,y)\xi \sin x\xi \, d\xi \Big|_{x=0} = 0$

Ekvationen $u_{xx} + u_{yy} = 0$ ska cos-transformeras!

$$\mathcal{F}_c u_{xx}(\xi,y) = \int_0^{+\infty} u_{xx}(x,y)\cos(x\xi)\,dx \overset{PI}{=}$$

$$= \underbrace{\Big[u_x \cos x\xi\Big]_0^\infty}_{=0} + \int_0^\infty u_x \cdot \xi \sin x\xi \, dx =$$

$$= \underbrace{\Big[u\,\xi \sin x\xi\Big]_0^\infty}_{=0} - \int_0^\infty u\,\xi^2 \cos x\xi \, dx =$$

$$= -\xi^2 w(\xi,y)$$

$$\mathcal{F}_c u_{yy} = w_{yy}(\xi,y)$$

Ekvationen blir:

$$-\xi^2 w(\xi,y) + w_{yy}(\xi,y) = 0$$

$w_y(\xi,0) = 0 \quad \forall \xi$ randvillkor

$w(\xi,1) = \mathcal{F}_c e^{-x}(\xi) = \frac{1}{1+\xi^2}$ {övn 7.4.1}

Ekv. ger: $w(\xi,y) = A(\xi)\exp(\xi y) + B(\xi)\exp(-\xi y)$ eller

ekvivalent: $w(\xi,y) = C(\xi)\cosh(\xi y) + D(\xi)\sinh(\xi y)$

~~$\omega(\xi,0)=0$ medför~~

$w_y(\xi,0)=0$ medför att $D(\xi)=0$

$$\left\{ w_y(\xi,y) = C(\xi)\xi \sinh \xi y + D(\xi)\xi \cosh \xi y \right\}$$

$$w(\xi,y) = C(\xi)\cosh \xi y$$

$w(\xi,1) = \frac{1}{1+\xi^2}$ medför $C(\xi)\cosh\xi = \frac{1}{1+\xi^2}$

$$w(\xi,y) = \frac{\cosh(\xi y)}{(1+\xi^2)\cosh\xi}$$

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\cosh \xi y}{(1+\xi^2)\cosh\xi} \cos x\xi \, d\xi$$

$\boxed{\text{Holmåkers Övn 2}}$

$\Omega T = 2\pi$, $\Omega > \alpha$

$T < \frac{2\pi}{\alpha}$, $\alpha T < 2\pi$

samplingssatsen:

$\alpha T \leq \pi \longrightarrow g = f$

$$g = LP_\alpha\left(T\sum_{n=-\infty}^{+\infty} f(nT)\,\delta(t-nT)\right)$$

$$\sum_{-\infty}^{+\infty} f(nT)\,\delta(t-nT) = \sum_{-\infty}^{+\infty} f(t)\,\delta(t-nT) = f(t)\sum_{-\infty}^{+\infty}\delta(t-nT)$$

$$\sum_{n=-\infty}^{+\infty}\delta(t-nT) = \sum_{k=-\infty}^{+\infty} c_k \exp\left(i\frac{2\pi}{T}kt\right)$$

$$c_k = \frac{1}{T} \quad \forall k$$

$$g = LP_\alpha\left(T f(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T} \exp\left(i \frac{2\pi}{T} kt\right)\right) =$$

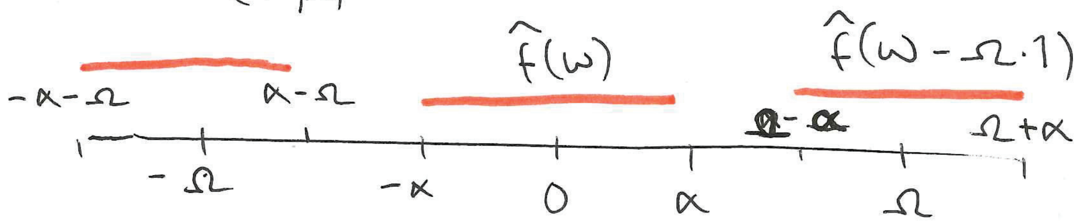
$$= LP_\alpha\left(\sum_{-\infty}^{+\infty} f(t) \exp\left(i \frac{2\pi}{T} kt\right)\right) =$$

$$= \mathcal{F}^{-1} \underbrace{}_{\chi_{(-\alpha,\alpha)}} \widehat{\sum_{-\infty}^{+\infty} f(t) \exp\left(i \frac{2\pi}{T} kt\right)} =$$

$$= \mathcal{F}^{-1} \chi_{(-\alpha,\alpha)} \sum_{-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2\pi}{T} k\right)$$

$$\hat{g}(\omega) = \chi_{(-\alpha,\alpha)}(\omega) \sum_{-\infty}^{+\infty} \hat{f}(\omega - \Omega k)$$

$$\hat{f}(\omega) = \chi_{(-\alpha,\alpha)}(\omega)$$



Om $+\Omega-\alpha \geq \alpha$, dvs $\Omega \geq 2\alpha$ blir intervallen disjunkta och $\hat{g}(\omega) = \chi_{(-\alpha,\alpha)}(\omega) = \hat{f}(\omega)$ och $g=f$

Men om $\Omega < 2\alpha$, men $\Omega > \alpha$

$$\hat{g}(\omega) = \begin{cases} 1 & |\omega| < \Omega - \alpha \\ 2 & \Omega - \alpha < |\omega| < \alpha \\ 0 & |\omega| > \alpha \end{cases}$$

$$\int_{-\infty}^{+\infty} |f(t) - g(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega) - \hat{g}(\omega)|^2 \, d\omega =$$

$$= \frac{1}{2\pi} \int\limits_{\Omega - \alpha < |\omega| < \alpha} 1 \, d\omega = \frac{1}{2\pi} \cdot 2(\alpha - (\Omega - \alpha)) = \frac{2\alpha - \Omega}{\pi}$$

EÖ20 N $\quad x(n) = \sin \frac{\pi n}{N}, \quad n = 0, \ldots, N-1$

$$\hat{x}_m = \sum_{n=0}^{N-1} \sin \frac{\pi n}{N} \exp\left(-i \frac{2\pi}{N} mn\right) =$$

$$= \frac{1}{2i} \sum_{n=0}^{N-1} \left(\exp\left(-i \frac{2\pi}{N}\left(m - \tfrac{1}{2}\right)n\right) - \exp\left(-i \frac{2\pi}{N}\left(m + \tfrac{1}{2}\right)n\right) \right)$$

$$\sum_{0}^{N-1} \exp\left(-i \frac{2\pi}{N}\left(m \pm \tfrac{1}{2}\right)n\right) = \frac{1 - \exp\left(-i \frac{2\pi}{N}\left(m \pm \frac{1}{2}\right)N\right)}{1 - \exp\left(-i \frac{2\pi}{N}\left(m \pm \frac{1}{2}\right)\right)}$$

$$\ulcorner \exp\left(i \frac{2\pi}{N}\left(m \pm \tfrac{1}{2}\right)N\right) = \exp(\pm i\pi) = -1 \lrcorner$$

$$\hat{x}_m = \frac{1}{2i} \cdot 2 \left(\frac{-1}{1 - \exp\left(-i \frac{2\pi}{N}\left(m + \frac{1}{2}\right)\right)} + \frac{1}{1 - \exp\left(-i \frac{2\pi}{N}\left(m - \frac{1}{2}\right)\right)} \right)$$

$$= \frac{1}{i} \frac{\exp\left(-i \frac{2\pi}{N}\left(m - \frac{1}{2}\right)\right) - \exp\left(-i \frac{2\pi}{N}\left(m + \frac{1}{2}\right)\right)}{\left(1 - \exp\left(-i \frac{2\pi}{N}\left(m + \frac{1}{2}\right)\right)\right)\left(1 - \exp\left(-i \frac{2\pi}{N}\left(m - \frac{1}{2}\right)\right)\right)} =$$

$$= \frac{\exp\left(-i \frac{2\pi}{N} m\right)\left[\exp\left(i \frac{\pi}{N}\right) - \exp\left(-i \frac{\pi}{N}\right)\right]}{i \left(\left[\exp\left(-i \frac{\pi}{N} m\right)\right]^2 \left[\exp\left(i \frac{\pi}{N} m\right) - \exp\left(-i \frac{\pi}{N}(m+1)\right)\right]\left(e^{i\frac{\pi}{N}m} - e^{-i\frac{\pi}{N}(m-1)}\right)\right)}$$

$$= \frac{1}{i} \frac{\exp\left(i \frac{\pi}{N}\right) - \exp\left(-i \frac{\pi}{N}\right)}{\exp\left(-i \frac{\pi}{2N}\right)\left[e^{i\frac{\pi}{N}\left(m + \frac{1}{2}\right)} - e^{-i\frac{\pi}{N}\left(m + \frac{1}{2}\right)}\right] e^{i\frac{\pi}{N}\frac{1}{2}} \left(e^{i\frac{\pi}{N}\left(m - \frac{1}{2}\right)} - e^{-i\frac{\pi}{N}\left(m - \frac{1}{2}\right)}\right)}$$

Nämnaren $= \exp(i \frac{2\pi}{N} m) + \exp(-i \frac{2\pi}{N} m) -$

$- \exp(-i \frac{\pi}{N}) - \exp(i \frac{\pi}{N})$

$$\hat{X}_m = \frac{\sin \frac{\pi}{N}}{\cos \frac{2\pi}{N} m - \cos \frac{\pi}{N}}$$