Forts från föreläsningen:

Mån LV6

$n = n'$, $k \neq k'$ $\longrightarrow$ $\int J_n\left(\frac{\lambda_{kn}}{b} r\right) J_n\left(\frac{\lambda_{k'n}}{b} r\right) r\,dr = 0$

3.5.10 $(xf')' + \lambda x^{-1} f = 0$ i $[1, b]$

$f(1) = f(b) = 0$

$r(x) = x$, $p(x) = 0$, $w(x) = \frac{1}{x}$

Reg. SL-problem!

$x f'' + f' + \frac{\lambda}{x} f = 0$

$x^2 f'' + x f' + \lambda f = 0$ Eulerekvation

Ansätt $f = x^{\gamma}$ $\longrightarrow$ $\gamma(\gamma - 1) + \gamma + \lambda = 0$ $\gamma^2 = -\lambda$

$\lambda < 0$

Sätt $\lambda = -\mu^2$, $\gamma = \pm \mu$

Allm. lösning: $f(x) = a x^{\mu} + b' x^{-\mu}$

RV: $a + b' = 0$ $\longrightarrow$ $f(x) = x^{\mu} - x^{-\mu}$

$f(b) = 0$ ger $b^{\mu} - b^{-\mu} = 0$ ingen lösning!

$\lambda = 0$

$f(x) = a + b' \ln x$

$f(1) = 0$ ger $a = 0$, $f(b) = 0$ ger $b = 0$ ingen lösning!

$\underline{\lambda > 0}$

sätt $\lambda = \mu^2$, $\gamma = \pm i\mu$

$f(x) = a x^{i\mu} + b' x^{-i\mu}$

$f(1) = 0$ ger $a + b' = 0 \rightarrow f(x) = x^{i\mu} - x^{-i\mu}$

$f(b) = 0$ ger $b^{i\mu} - b^{-i\mu} = 0$

$f(x) = \exp(i\mu \ln x) - \exp(-i\mu \ln x) = 2i \sin(\mu \ln x)$

ta $f(x) = \sin(\mu \ln x)$ Kastar $2i$

$f(b) = 0 \rightarrow \sin(\mu \ln b) = 0$

$$\mu = \frac{n\pi}{\ln b} \qquad n = 1, 2, \ldots$$

Egenvärden: $\lambda = \left(\frac{n\pi}{\ln b}\right)^2$, $n = 1, 2, \ldots$

Egenfunktioner: $\sin\left(\frac{n\pi}{\ln b} \ln x\right)$

$$\left\| \sin\left(\frac{n\pi}{\ln b} \ln x\right) \right\|_w^2 = \int_1^b \sin^2\left(\frac{n\pi}{\ln b} \ln x\right) \underbrace{\frac{1}{x}}_{\text{vikten, } w(x)} dx =$$

$$= \left\{ \begin{array}{l} \ln x = s \\ ds = \frac{1}{x} dx \end{array} \right\} = \int_0^{\ln b} \sin^2\left(\frac{n\pi}{\ln b} s\right) ds = \frac{1}{2} \int_0^{\ln b} \left(1 - \cos\left(\frac{2\pi n}{\ln b} s\right)\right) ds =$$

$$= \frac{\ln b}{2} - \left[\frac{\ln b}{4\pi n} \sin\left(\frac{2\pi n}{\ln b} s\right)\right]_0^{\ln b} = \frac{\ln b}{2}$$

Normaliserade egenfunktioner:

$$\sqrt{\frac{2}{\ln b}} \sin\left(\frac{n\pi}{\ln b} \ln x\right)$$

$$g(x) = 1 \quad ; \quad g(x) = \sum_{1}^{n} c_n \sin\left(\frac{\pi n}{\ln b} \ln x\right)$$

där $$c_n = \frac{2}{\ln b} \int_1^b 1 \cdot \sin\left(\frac{\pi n}{\ln b} \ln x\right) \frac{dx}{x} =$$

$$= \frac{2}{\ln b} \int_0^{\ln b} \sin\left(\frac{\pi n}{\ln b} s\right) ds = \frac{2}{\ln b} \frac{\ln b}{\pi n} \left[-\cos\left(\frac{\pi n}{\ln b} s\right)\right]_0^{\ln b} =$$

samma slost. som för!

$$= \frac{2}{\pi n}\left(1-(-1)^n\right) = \begin{cases} \frac{4}{\pi n} & \text{då } n \text{ udda} \\ 0 & \text{då } n \text{ jämn} \end{cases}$$

$$g(x) = 1 = \sum \frac{4}{\pi(2k-1)} \cdot \cancel{\sin} \sin\left(\frac{\pi(2k-1)}{b}\right)$$

$$f_6(s) = 1 = \frac{4}{\pi} \sum_{1}^{\infty} \frac{\sin(2k-1)}{2k-1}$$

$0 < \theta < \pi$

$$\theta = \frac{\pi}{\ln b} = \frac{\pi}{\ln b} = 1$$

4.2.4

$$u_t = k u_{xx} + R \qquad 0 < x < L$$

$$u(0,t) = 0 \,,\; u_x(L,t) = 0$$

$$u(x,0) = f(x)$$

steady-state: Sök $u_0 = u_0(x)$ s.a.

$$0 = k(u_0)_{xx} + R \quad \text{och} \quad u_0(0) = 0 \,,\; u_0'(L) = 0$$

$$0 = k u_0''(x) + R \qquad u_0'' = -\frac{R}{k}$$

$$u_0(x) = -\frac{R}{2k}x^2 + ax + b$$

$u_0(0) = 0$ ger $b = 0$,

$u_0'(L) = 0$ ger $-\frac{R}{k}L + a = 0 \implies a = \frac{RL}{k}$

$$u_0(x) = -\frac{R}{2k}x^2 + \frac{RL}{k}x = \frac{R}{2k}x(2L - x)$$

Sök lösning u av form

$$u(x,t) = v(x,t) + u_0(x,t)$$

~~För~~

$u_t = k u_{xx} + R$ ger

$$v_t = k v_{xx} + \underbrace{k(u_0)_{xx} + R}_{=0} \qquad \begin{cases} v(0,t) = 0 \\ v_x(L,t) = 0 \end{cases}$$

Var.sep. $v = X T \qquad X'' = \lambda X \qquad T' = k\lambda T$

$X(0) = 0 \qquad X'(L) = 0 \qquad$ Får som tidigare

$$X = \sin\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}x \quad ; \quad \lambda = -\left[\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}\right]^2$$

$$T(t) = \exp\left[-k\left[\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}\right]^2 t\right]$$

Begynnelsevillkor: $v(x,0) = f(x) - U_0(x)$

Ansätt $v(x,t) = \sum_{1}^{\infty} b_n \sin\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}x \exp\left[-k\left(\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}\right)^2 t\right]$

$t=0$: $\sum b_n \sin\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}x = f(x) - \frac{R}{2k}x(2L-x)$

b_n ges som sinus-koeff. för $f(x) - \frac{R}{2k}x(2L-x)$

Sen $u(x,t) = U_0(x) + \sum b_n \ldots$

Utveckla $x(2L-x)$ i $\sin\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}x$-serie

koeff.: $\frac{2}{L}\int_0^L x(2L-x)\sin\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}x\,dx = \frac{8L^2}{\pi^3}\frac{1}{\left(n-\tfrac{1}{2}\right)^3}$

(f_{17}) BETA 13.1 (7)

$$t(L-t) = \frac{8L^2}{\pi^3}\sum \frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi}{L}t$$

$$L = 2\ell \longrightarrow t(2\ell-t) = \frac{32\ell^2}{\pi^3}\sum_{1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{\left(n-\tfrac{1}{2}\right)\pi}{\ell}t$$

EÖ 24 $-\exp(4x)\dfrac{d}{dx}\left(\exp(4x)\dfrac{du}{dx}\right) = \lambda u \qquad 0<x<1$

$u(0)=0 \quad , \quad u'(1)=0$

$(e^{4x}u')' + \lambda e^{4x}u = 0$

Reg. SL-problem ; $r(x) = e^{4x}$, $w(x) = e^{4x}$

$u'' + 4u' + \lambda u = 0$

Kar. ekv. : $r^2 + 4r + \lambda = 0$

$r = -2 \pm \sqrt{4-\lambda}$

Fall 1: $\lambda < 4$

Sätt $\lambda = 4 - \mu^2 \Rightarrow 4 - \lambda = \mu^2$

$r = -2 \pm \mu \Rightarrow u = a\exp\big((-2+\mu)x\big) + b\exp\big((-2-\mu)x\big)$

$u(0)=0$ ger $a+b=0$

$\Rightarrow u = \exp\big((-2+\mu)x\big) - \exp\big((-2-\mu)x\big) =$

$= 2e^{-2x}\sinh \mu x$ Kasta 2

$u'(1)=0$ ger $-2e^{-2x}\sinh\mu x + \mu e^{-2x}\cosh\mu x = 0$

$\Rightarrow \tanh\mu = \dfrac{\mu}{2}$

$\tanh' x = \dfrac{1}{\cosh^2 x}$

En lösning μ_0

Egenvärde: $\lambda_0 = 4 - \mu_0^2$

Egenfunktion: $\varphi_0(x) = e^{-2x} \sinh \mu_0 x$

$\lambda = 4 \longrightarrow r = -2$ (dubbelrot)

$u(x) = a e^{-2x} + b x e^{-2x}$

$u(0) = 0$ ger $a = 0 \longrightarrow u(x) = x e^{-2x}$

$u'(1) = 0$ ger $e^{-2x} - 2x e^{-2x} = 0$ för $x = 1$

Ingen lösning!

$\lambda > 4 \longrightarrow \lambda = 4 + \mu^2 \qquad \mu^2 = \lambda - 4$

$r = -2 \pm i\mu$

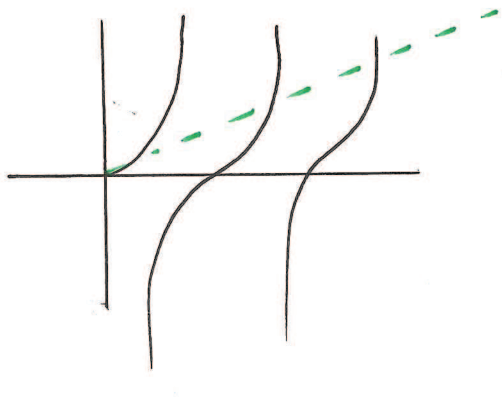
$u(x) = e^{-2x}(a \cos \mu x + b \sin \mu x)$

$u(0) = 0$ ger $a = 0 \longrightarrow u(x) = e^{-2x} \sin \mu x$

$u'(1) = 0$: $-2e^{-2x} \sin \mu x + \mu e^{-2x} \cos \mu x = 0$ för $x = 1$

$\tan \mu = \dfrac{\mu}{2}$

Lösning $\mu_1, \mu_2, \ldots$



Egenvärden: $\lambda_k = 4 + \mu_k^2$

Egenfunktioner: $e^{-2x} \sin \mu_k x$, som bildar ett fullständigt ortogonalsystem i $L^2_w[0,1]$

$$\varphi_k(x) = e^{-2x} \sin \mu_k x$$

$$\|\varphi_k\|_w^2 = \int_0^1 (e^{-2x} \sin \mu_k x)^2 e^{4x} \, dx = \int_0^1 \sin^2 \mu_k x \, dx =$$

$$= \frac{1}{2} - \frac{\sin 2\mu_k}{4\mu_k} = \frac{\lambda_k - 2}{2\lambda_k}$$

$$e^{-2x} = \sum_{k=0}^{\infty} c_k \varphi_k$$

$$c_k = \frac{1}{\|\varphi_k\|_w^2} \int_0^1 \underbrace{e^{-2x} e^{-2x} \sin \mu_k x \, e^{4x}}_{= \sin \mu_k x} \, dx$$

$k \geq 1$

c_0 analogt