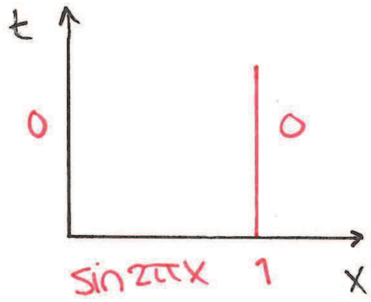
Mån Lv 7

EÖ28 $\quad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = t \sin x \qquad 0<x<1, \quad t>0$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(2\pi x)$$



"Kan vi använda steady-state?" NEJ! !!

Variabla koefficienter {Metod/teknik 2}

Ansätt $\displaystyle u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \pi n x$

Ekv. blir $\cancel{\displaystyle \sum_{n=1}^{\infty} \left(b_n'(t) - \pi^2 n^2\right) \sin \pi n x = t \sin x}$

$$\sum_{n=1}^{\infty} \left(b_n'(t) + \pi^2 n^2 b(t)\right) \sin \pi n x = t \sin x$$

Utveckla $\displaystyle \sin x = \sum_{n=1}^{\infty} \beta_n \sin \pi n x \qquad 0<x<1$

$$\beta_n = \frac{2}{L} \int_0^L \sin x \sin \pi n x \, dx = \left\{ L \equiv 1 \right\} =$$

$$= \frac{2}{1} \int_0^1 \sin x \sin \pi n x \, dx = \int_0^1 \left(\cos(\pi n - 1)x - \cos(\pi n + 1)x\right) dx$$

$$= \left[\frac{\sin(\pi n - 1)x}{\pi n - 1} - \frac{\sin(\pi n + 1)x}{\pi n + 1} \right]_0^1 =$$

$$= \frac{\sin(\pi n - 1)}{(\pi n - 1)} - \frac{\sin(\pi n + 1)x}{(\pi n + 1)} =$$

$$= -(-1)^n \sin 1 \left(\frac{1}{\pi n - 1} + \frac{1}{\pi n + 1} \right) = (-1)^{n+1} \sin 1 \, \frac{2\pi n}{\pi^2 n^2 - 1}$$

Alt. Titta i BETA 13.1 (16)

$$\sin \alpha t = \frac{2 \sin \alpha \pi}{\pi} \sum_1^{\infty} (-1)^{n+1} \frac{n}{n^2 - \alpha^2} \sin nt \qquad |t| < \pi, \quad \alpha \notin \mathbb{Z}$$

Ta $t = \pi x$ $\sin \pi \cancel{\alpha} x$, ta $\alpha = \frac{1}{\pi}$

Identifiering av koefficienten ger:

$$b_n'(t) + \pi^2 n^2 b_n(t) = t \beta_n$$

$$t = 0 \implies \sum_{n=1}^{\infty} b_n(0) \sin \pi n x = \sin 2\pi x$$

$$b_2(0) = 1 \qquad b_n(0) = 0, \quad n \neq 2$$

Multipl. med integrerande faktorn $\exp(\pi^2 n^2 t)$

$$\left(\exp(\pi^2 n^2 t) b_n(t)\right)' = t \beta_n \exp(\pi^2 n^2 t)$$

$$\exp(\pi^2 n^2 t) b_n(t) = \exp(\pi^2 n^2 t) b_n(0) + \int_0^t \beta_n t' e^{\pi^2 n^2 t'} \, dt'$$

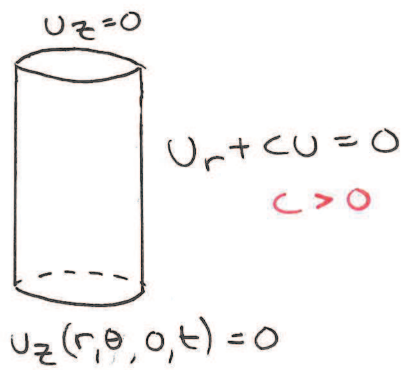
$$= b_n(t) = \beta_n \left(\frac{1}{\pi^2 n^2} t - \frac{1}{n^4 \pi^4} + \frac{1}{n^4 \pi^4} \exp(-\pi^2 n^2 t) \right) \quad n \neq 2$$

$$b_2(t) = \exp(-4\pi^2 t) + \beta_2 \left(\frac{1}{4\pi^2} t - \frac{1}{16\pi^4} + \frac{1}{16\pi^4} \exp(-4\pi^2 t) \right)$$

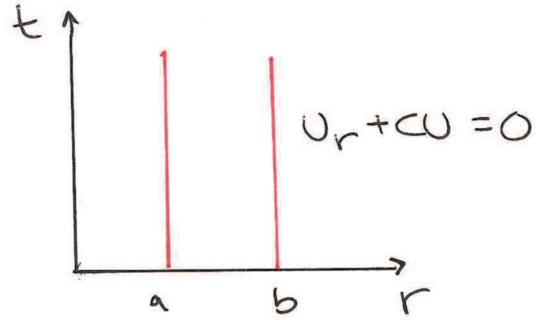
5.5.1

$u_t = k\Delta u$

$u = u(r, \theta, z, t)$



u är oberoende av θ och av z, ty lösningen till motsvarande problem utan z, i tvårumsdim., löser även det givna problemet



$u = u(r, t)$

$u(r, 0) = A$

Variabelseparation: $u = R(r)T(t)$

$$R(r)T'(t) = k\left(R''T + \frac{1}{r}R'T\right) \qquad \begin{cases} R = R(r) \\ T = T(t) \end{cases}$$

$$\frac{1}{k}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \lambda \quad \text{konst.}$$

$$R'' + \frac{1}{r}R' - \lambda R = 0 \quad \Longrightarrow \text{Bessels!}$$

$$r^2R'' + rR' - \lambda r^2 R = 0$$

{ Bessel, eller mod. Bessel med $\nu = 0$ }

$\lambda > 0$: $\lambda = \mu^2$, $\mu > 0$

$$r^2R'' + rR' - \mu^2r^2R = 0 \quad \text{mod. Bessel ekv.}$$

Lösn. $R(r) = a\,I_0(\mu r) + b'\,K_0(\mu r)$

$b' = 0$ ty K_0 singulär vid 0

Men $I_0, I_0' > 0$ så $I_0(\mu r)$ kan ej uppfylla randvillkor!

$I_0(\mu r)$ kan ej uppfylla randvillkoret $R'(b) + cR(b) = 0$

$\Rightarrow$ Inga lösn. för $\lambda > 0$

$\lambda = 0$ $\quad R(r) = a + b' \ln r$

$b' = 0, \quad a = 0$ $\quad$ pss som för $\lambda > 0$

$\lambda < 0$ $\quad \lambda = -\mu^2, \quad \mu > 0$

$r^2 R'' + rR' + \mu^2 r^2 R = 0$ $\qquad$ Bessels ekv.

Lösn. $\quad R(r) = a J_0(\mu r) + b' Y_0(\mu r)$

$b' = 0$ ty Y_0 singulär i 0

$R(r) = J_0(\mu r)$ skall satisfiera $R'(b) + c R(b) = 0$

dvs $\quad \mu J_0'(\mu b) + c J_0(\mu b) = 0$

$c' J_0(x) + x J_0'(x) = 0$

har nollst. $\tilde{\lambda}_k$ $\qquad$ Eulers sats

Så μb skall vara ett nollställe till

$$x J_0'(x) + bc J_0(x)$$

Alltså $\quad \mu b = \tilde{\lambda}_k$ $\quad$ något $k \in \{1, 2, \ldots\}$

$$\mu = \frac{\tilde{\lambda}_k}{b}$$

För T fås

$$T' = -k\left(\frac{\tilde{\lambda}_k}{b}\right)^2 T$$

$$T = c\exp\left(-\tilde{k}\left(\frac{\tilde{\lambda}_k}{b}\right)^2 t\right)$$

sep. lösn. : $J_0\left(\frac{\tilde{\lambda}_k}{b} r\right)\exp\left(-\tilde{k}\left(\frac{\tilde{\lambda}_k}{b}\right)^2 t\right)$

Ansätt

$$u(r,t) = \sum_{k=}^{\infty} c_k J_0\left(\frac{\tilde{\lambda}_k}{b} r\right)\exp\left(-\tilde{k}\left(\frac{\tilde{\lambda}_k}{b}\right)^2 t\right)$$

$u(r,0) = A$ ger

$$\sum_{k=1}^{\infty} c_k J_0\left(\frac{\tilde{\lambda}_k}{b} r\right) = A \qquad 0<r<b$$

$$c_k = \frac{1}{\left\|J_0\left(\frac{\tilde{\lambda}_k}{b} r\right)\right\|_w^2}\int_0^b A J_0\left(\frac{\tilde{\lambda}_k}{b} r\right) r\,dr =$$

$\{$ där $w(r) = r$ i $[0,r]$ $\}$

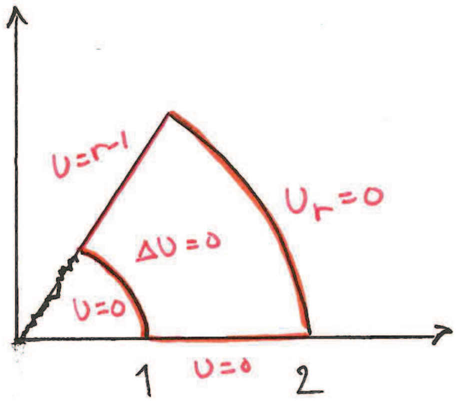
$$= A\frac{2\lambda_k}{b^2\left(\lambda_k^2 + (c')^2 b^2\ J_0\left(\frac{\lambda_k}{}\right)^2\right)}\left(\frac{b}{\lambda_k}\right)^2 \cdot \underbrace{\int_0^{\lambda_k} J_0(s)\, s\, ds}_{\tilde{\lambda}_k J_1(\tilde{\lambda}_k)}$$

$\{c' = bc\}$

ehligt övn. 5.2.4

EÖ301

$\Delta U = 0$



Var. sep:

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda \quad \text{konst.}$$

$$r^2 R'' + rR' - \lambda R = 0$$

Eulerekv. : ansats r^γ ger $\gamma(\gamma-1) + \gamma - \lambda = 0$

$$\gamma = \pm\sqrt{\lambda}$$

$\lambda > 0$ $\lambda = \mu^2$ ger allm. lösning:

$$ar^{\mu} + br^{-\mu} = R(r)$$

$R(1) = 0$ ger $a + b = 0$, $R(r) = a(r^{\mu} - r^{-\mu})$

$$R'(r) = a\mu(r^{\mu-1} + r^{-\mu-1}) = \frac{a\mu}{r}(r^{\mu} + r^{-\mu}) > 0$$

Randvillkor $R'(2) = 0$ går ej!

$\lambda = 0$ $R(r) = a + b\ln r$. Randvillkor ger

$a = 0$, $\frac{b}{r} = 0$ för $r = 2$

Ingen lösning!

$\lambda<0$ $\quad \lambda = -\mu^2, \ \mu > 0$ $\quad\quad$ {obs! $x \equiv r$}

Allmän lösning är

$$R(r) = ar^{i\mu} + br^{-i\mu} = ae^{i\mu \ln x} + be^{-i\mu \ln x}$$

$R(1)$ ger $R(1) = 0 \longrightarrow a+b=0$

$$R(r) = a \cdot 2i \, \cancel{\sin h} \sin(\mu \ln r)$$

$$R'(r) = \mu \cdot \frac{1}{r} \cos(\mu \ln r)$$

$R'(2) = 0$ ger $\cos(\mu \ln 2) = 0$

$$\mu \ln 2 = \left(n - \frac{1}{2}\right)\pi \quad\quad n = 1, 2, \ldots$$

$$\Longrightarrow \mu = \left(n - \frac{1}{2}\right)\frac{\pi}{\ln 2}$$

$$\Theta'' = \mu^2 \Theta = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{\ln 2}\right]^2 \Theta$$

$\Theta(0) = 0$ så $\Theta(\theta) = \sinh\left(\left(n - \frac{1}{2}\right)\frac{\pi}{\ln 2}\theta\right)$