Ö431 $\Delta U = 0$, $r < R_0$

Fre Lv 7

$U = U(r, \theta, \varphi)$ $U = z(x^2 + y^2)$ för $r = R_0$

Obs!

$z = r\cos\theta$

$y = r\sin\theta\sin\varphi$

$x = r\sin\theta\cos\varphi$

$$U(R_0, \theta, \varphi) = R_0\cos\theta\left(R_0^2\sin^2\theta\cos^2\varphi + R_0^2\sin^2\theta\sin^2\varphi\right)$$

$$= R_0\cos\theta\left(R_0^2\sin^2\theta\left(\cos^2\varphi + \sin^2\varphi\right)\right)$$

$$= R_0^3\cos\theta\sin^2\theta$$

U oberoende av φ

vet att var. sep. ger:

$$U(r, \theta, \varphi) = \sum_{n=0}^{\infty}\sum_{|m| \leq n} c_{nm} r^n P_n^{|m|}(\cos\theta) e^{im\varphi}$$ {Finns EJ i BETA}

om oberoende av φ:

$$U(r, \theta) = \sum_{n=0}^{\infty} r^n c_n P_n(\cos\theta)$$ {Finns i BETA}

Randvillkoret ger:

$$U(R_0, \theta) = \sum_{n=0}^{\infty} c_n R_0^n P_n(\cos\theta) = R_0^3\cos\theta\sin^2\theta$$

$-\pi \leq \theta \leq \pi$

$$\left\{ \sin^2\theta = 1-\cos^2\theta \right\}$$

Men $s = \cos\theta$:

$$\sum_{n=0}^{\infty} c_n R_0^n P_n(s) = R_0^3 s(1-s^2)$$

$$\rightarrow \sum_{n=0}^{\infty} c_n R_0^{n-3} P_n(s) = s - s^3$$

$$c_n R_0^{n-3} = \frac{1}{\| P_n \|^2_{(-1,1)}} \int_{-1}^{1} (s-s^3) P_n(s)\, ds =$$

Legendre polynom (ur BETA :))♡

$$P_0 = 1 \qquad P_1 = s \qquad P_2 = \frac{3}{2}s^2 - \frac{1}{2} \qquad P_3 = \frac{5}{2}s^2 - \frac{3}{2}s$$

$$s = P_1(s), \quad s^3 = \frac{2}{5}P_3(s) + \frac{3}{5}P_1(s)$$

$$\Longrightarrow s - s^3 = \frac{2}{5}P_1(s) - \frac{2}{5}P_3(s)$$

$$n=1 \rightarrow c_1 R_0^{-2} = \frac{2}{5}$$

$$n=3 \rightarrow c_3 = -\frac{2}{5}$$

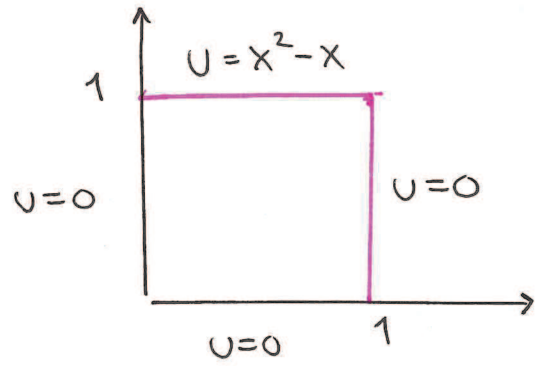
$$U(r,\theta) = \frac{2}{5}R_0^2 r \cancel{\cos} P_1(\cos\theta) - \frac{2}{5}r^3 P_3(\cos\theta) =$$

$$= \frac{2}{5}R_0^2 r\cos\theta - \frac{2}{5}r^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) =$$

$$= \frac{2}{5}R_0^2 \cdot z - z^3 + \frac{3}{5}\underbrace{r^3\cos\theta}_{= z(x^2+y^2+z^2)}$$

Eö 27 $U''_{xx} + U''_{yy} + 20U = 0$ $\quad 0 < x, y < 1$

kvadrat med längd 1



Homogen!

Ty $20U = f(U)$

Börja med att var. separera: $U = X(x)Y(y)$

$$\frac{X''}{X} + \frac{Y''}{Y} + 20 = 0$$

$$\Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} + 20 = \lambda \quad \text{konst.}$$

classic! :)

$X(0) = 0, \quad X(1) = 0$ Randvillkor, $0 < x < 1$

$X(x) = \sin n\pi x \quad n = 1, 2, \ldots$

Ger $\lambda = \mu^2 = n^2\pi^2 = (n\pi)^2$

För Y fås

$$Y'' + 20Y = (n\pi)^2 Y \longrightarrow Y'' = Y\left((n\pi)^2 - 20\right)$$

$Y(0) = 0,$

$$(n\pi)^2 - 20 = \begin{cases} < 0 & n = 1 \\ > 0 & n > 1 \end{cases}$$

$\underline{n=1}$ $\quad Y = a\cos\left(\sqrt{20-\pi^2}\,y\right) + b\sin\left(\sqrt{20-\pi^2}\,y\right)$

$Y(0)=0$ ger $a=0$ $\longrightarrow$ $Y = \sin\left(\sqrt{20-\pi^2}\,y\right)$

$\underline{n>1}$ $\quad Y = a\cosh\sqrt{(n\pi)^2-20}\;y + b\sinh\sqrt{(n\pi)^2-20}\;y$

$Y(0)=0$ ger $a=0$ $\longrightarrow$ $Y = \sinh\sqrt{(n\pi)^2-20}\;y$

Ansätt

$$u = a_1 \sin \pi x \sin\sqrt{20-\pi^2}\,y + \sum_{n=2}^{\infty} a_n \sin n\pi x \sinh\sqrt{(n\pi)^2-20}\,y$$

$\underline{y=1}$ $\quad a_1 \sin\pi x\sqrt{20-\pi^2} + \sum_{2}^{\infty} a_n \sin n\pi x \sinh\sqrt{(n\pi)^2-20}$

$= x^2 - x$ Randvillkor

Utveckla $x^2-x = -x(1-x) = \ldots$ i sin-serie

BETA 13.2 (7): $x(L-x)$ med $L=1$

EÖ 36 $\min \int_0^\infty |\sqrt{x} - P(x)|^2 e^{-x}\, dx$

P polynom grad ≤ 2

Laguere, vikt $x^\alpha e^{-x}$ i $(0,+\infty)$, ta $\alpha = 0$

$L_n^0(x) = L_n(x)$ $n = 0, 1, \ldots$ är ortogonala

Bästa approx.-satsen säger att

$$P(x) = \sum_{n=0}^{2} c_n L_n \quad , \quad c_n = \frac{1}{\| L_n \|_{e^{-x}}} \int_0^\infty \sqrt{x}\, L_n(x) e^{-x}\, dx$$

$$\| L_n \|^2 = \frac{\Gamma(n+1)}{n!} = 1$$

$L_0 = 1$ $\quad L_1 = 1 - x$ $\quad L_2 = \frac{x^2}{2} - 2x + 1$ BETA ❤

Behöver $\underbrace{\int_0^\infty x^{\frac{1}{2}} e^{-x}\, dx}_{(1)}$, $\underbrace{\int_0^\infty x^{\frac{3}{2}} e^{-x}\, dx}_{(2)}$, $\underbrace{\int_0^\infty x^{\frac{5}{2}} e^{-x}\, dx}_{(3)}$

Antingen $\sqrt{x} = y \Rightarrow \int_0^\infty y^k e^{-y^2}\, dy$

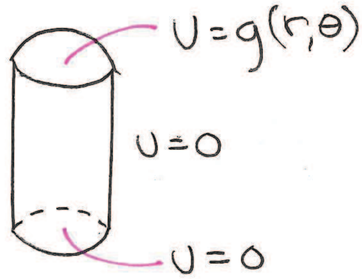
eller $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx \Rightarrow \begin{cases} (1) = \Gamma(\frac{3}{2}) \\ (2) = \Gamma(\frac{5}{2}) \\ (3) = \Gamma(\frac{7}{2}) \end{cases}$

Rekursionsformeln : $\Gamma(z+1) = z\Gamma(z)$ { BETA ❤ }

$\hookrightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$

5.5.5

$\Delta U = 0$



$U = U(r, \theta, z)$

$$\begin{cases} U(r,\theta,0) = 0 \\ U(b,\theta,z) = 0 \\ U(r,\theta,L) = g(r,\theta) \end{cases}$$

$U = R(r)\,\Theta(\theta)\,Z(z) \qquad R(b) = 0$

$$\rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = \lambda$$

$$\frac{r^2R'' + rR'}{R} - \lambda r^2 = -\frac{\Theta''}{\Theta} = n^2$$

$\Theta(\theta) = a\cos n\theta + b\sin n\theta \qquad n = 0, 1, 2, \ldots$

$r^2R'' + rR' - (\lambda r^2 + n^2)R = 0$

$\lambda > 0$: $\lambda = \mu^2$, $\mu > 0$ $\longrightarrow$ Mod. Bessel

$R(r) = c\,I_n(\mu r) + d\,K_n(\mu r)$

$d = 0$ ty K_n sing. i 0

$R(b) = 0$ medför $c = 0$ ty $I_n > 0$ på $\mathbb{R}_+$

Inga lösningar!

$\lambda = 0$: Eulerekv. $r^2R'' + rR' - n^2R = 0$

$R(r) = r^\gamma$ ger $\gamma(\gamma - 1) + \gamma - n^2 = 0$, $\gamma = \pm n$

$R = cr^n + dr^{-n}$, $n \neq 0$

$d = 0$ ty $r = 0$

$c = 0$ ty $R(b) = 0$

Inga lösningar!

$\lambda < 0$: $\lambda = -\mu^2$, $\mu > 0$ ger Bessels ekv.

$$R(r) = c J_n(\mu r) + d Y_n(\mu r)$$

$d = 0$ ty $r = 0$ $\longrightarrow$ $R(r) = J_n(\mu r)$

$R(b) = 0$ $\longrightarrow$ $\mu b = \lambda_{kn}$ $k = 1, 2, \ldots$

$(\lambda_{kn})_{k=1}^{\infty}$ är de positiva nollställena till J_n

$\Longrightarrow$ Får $\lambda = -\left(\frac{\lambda_{kn}}{b}\right)^2$

För Z fås $Z'' = \left(\frac{\lambda_{kn}}{b}\right)^2 Z$

$Z(0) = 0$ $\qquad Z = \alpha \cosh \frac{\lambda_{kn}}{b} z + \beta \sinh \frac{\lambda_{kn}}{b} z$

$\alpha = 0$ $\longrightarrow$ $Z = \sinh \frac{\lambda_{kn}}{b} z$

Sep. lösn. $\quad J_n\left(\frac{\lambda_{kn}}{b} r\right)(a \cos n\theta + b \sin n\theta) \sinh \frac{\lambda_{kn}}{b} z$

Ansätt

$$u(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\lambda_{kn}}{b} r\right)\left(a_{kn} \cos n\theta + b_{kn} \sin n\theta\right) \sinh \frac{\lambda_{kn}}{b} z$$

$z = L \Longrightarrow \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} J_n\left(\frac{\lambda_{kn}}{b} r\right)\left(a_{kn} \cos n\theta + b_{kn} \sin n\theta\right) \sinh \frac{\lambda_{kn}}{b} L = g(r, \theta)$

Utveckla $g(r, \theta)$ i ortogonalsystemet $J_n\left(\frac{\lambda_{kn}}{b} r\right) \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$

i $L^2\left((0, b) \times [-\pi, \pi]; r\, dr\, d\theta\right)$

$$a_{kn} \sinh \frac{\lambda_{kn}}{b} L = \frac{1}{\left\| J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos\theta \right\|_{L^2}^2} \iint_{\substack{0 < r < b \\ |\theta| \leq \pi}} g(r, \theta) J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos n\theta \; r\, dr\, d\theta$$