Mån LV2

$L^1 = \{ f ; \int |f(x)| dx \}$

$L^2 = \{ f ; \int |f(x)|^2 dx \}$

Faltning (konvolution)

Def. om $f, g \in L^1(\mathbb{R})$ så är $\overset{\text{faltning}}{f * g} = \int_{\mathbb{R}} f(x-y) g(y) dy$

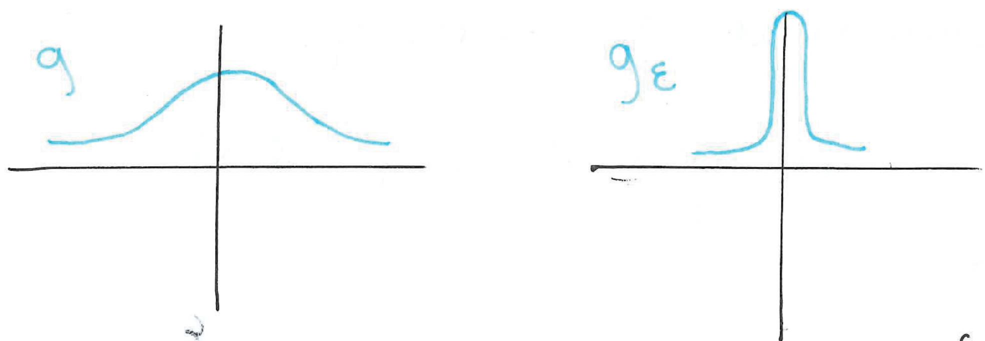
Variabelsubst. : $y' = x - y$

$\longrightarrow f * g = \int_{\mathbb{R}} f(x-y) g(y) dy \overset{\text{v.s.}}{=} \int f(y) g(x-y) dy$

Obs! $f * g = g * f$

Antag $g \in L^1(\mathbb{R})$, $\int g(y) dy = 1$ {ofta $g \geq 0$ men ej nödvändigt att anta}

sätt $g_\varepsilon(x) = \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right)$:



om $\int g(y) dy = 1$ och $g_\varepsilon(x) = \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right) \Longrightarrow \int g_\varepsilon(x) dx = 1$

Fråga: $f * g_\varepsilon(x) \rightarrow f(x)$ då $\varepsilon \rightarrow 0$?

Sats 7.3

Antag $g \in L^1(\mathbb{R})$. g är begränsad; $x^2 g(x) \to 0$ då $x \to \pm\infty$
$(g(x) = o(x^2))$

$\int g(x)dx = 1$. Om $f \in L^1$ är kontinuerlig i x
gäller:

$$f * g_\varepsilon(x) \to f(x), \quad \varepsilon \to 0$$

Om $f \in L^1(\mathbb{R})$ har ändliga vänster- och höger-
gränsvärden i x gäller:

$$\cancel{f * g_\varepsilon(x) \to \frac{1}{2}[f(-x) + f(x)]}$$

$$f * g_\varepsilon(x) \to \frac{1}{2}[f(x-) + f(x+)]$$

om dessutom g är jämn!

Anmärkning.

δ - (Dirac) - "funktionen" $\int_{\mathbb{R}} f(x)\delta(x)dx = f(0)$

för alla kontinuerliga f. Finns ej!

skulle få $f * \delta(x) = \int f(x-y)\delta(y)dy = f(x)$

dvs. $f * \delta = f$

(Satsen) $f * g_\varepsilon \to f * \delta$, $g_\varepsilon \to \delta$ om $\varepsilon \to 0$?

Bevis av satsen i en kontinuitetspunkt x

$$f * g_\varepsilon(x) - f(x) =$$

$$= \int f(x-y)\frac{1}{\varepsilon} g\left(\frac{y}{\varepsilon}\right)dy - f(x) = \left\{ \begin{array}{l} \frac{y}{\varepsilon} = z \\ \frac{1}{\varepsilon}dy = dz \end{array} \right\}$$

$$\cancel{\int f(z-\varepsilon)} \quad \int f(x-\varepsilon z)g(z)\,dz - \int f(x)g(z)\,dz =$$

$$= \int \big(f(x-\varepsilon z) - f(x)\big) g(z)\,dz$$

$$= \int_{|z|<\frac{1}{\sqrt{\varepsilon}}} \big(f(x-\varepsilon z) - f(x)\big)g(z)\,dz + \int_{|z|>\frac{1}{\sqrt{\varepsilon}}} f(x-\varepsilon z)g(z)\,dz - f(x)\int_{|z|>\frac{1}{\sqrt{\varepsilon}}} g(z)\,dz$$

$$= I_1 + I_2 + I_3$$

$$|I_1| \leq \int |f(x-\varepsilon z) - f(x)||g(z)|\,dz \leq \underbrace{\sup_{|y|<\sqrt{\varepsilon}} |f(x-y) - f(x)|}_{\to 0,\ \varepsilon \to 0} \underbrace{\int |g(z)|\,dz}_{<\infty} \to 0$$

$$|I_2| \leq \int |f(x-\varepsilon z)|\,dz \sup_{z>\frac{1}{\sqrt{\varepsilon}}} |g(z)| = \{\varepsilon z = y\} =$$

$$= \frac{1}{\varepsilon}\underbrace{\int |f(x-y)|\,dy}_{=\int|f|<\infty} \cdot \underbrace{\sup_{|z|>\frac{1}{\sqrt{\varepsilon}}} |g(z)|}_{=o\left(\left(\frac{1}{\sqrt{\varepsilon}}\right)^{-2}\right) = o(\varepsilon)} \to 0$$

{ ordo – en funktion som går mot noll, gånger det som står innanför parantesen }

Alt. $$\frac{1}{\varepsilon}\sup_{|z|>\frac{1}{\sqrt{\varepsilon}}}|g(z)| \leq \frac{1}{\varepsilon}\sup_{|z|>\frac{1}{\sqrt{\varepsilon}}} \underbrace{\varepsilon z^2}_{\geq 1}|g(z)| =$$

$$= \cancel{\frac{\varepsilon}{\varepsilon}} \sup_{|z|>\frac{1}{\sqrt{\varepsilon}}} z^2|g(z)| \longrightarrow 0$$

$$|I_3| \leq |f(x)| \int_{|z|>\frac{1}{\sqrt{\varepsilon}}} |g(z)|\,dz \longrightarrow 0$$

Fouriertransformen

Def. Ta $f \in L^1(\mathbb{R})$

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int f(x)e^{-ix\xi}\,dx, \quad \xi \in \mathbb{R}$$

$f \mapsto \hat{f}$

$$|\hat{f}(\xi)| \leq \int |f(x)|\,dx < \infty, \quad \hat{f} \text{ är begränsad}$$

Kan visas att $\hat{f}$ är kontinuerlig

Ex. $f = \chi_{(-a,a)}$ karakteristiskt

dvs $$f(x) = \begin{cases} 1, & x \in (-a,a) \\ 0, & \text{annars} \end{cases}$$

$$\hat{f}(\xi) = \int_{-a}^{a} 1 \cdot e^{-ix\xi}\,dx = \left[-\frac{1}{i\xi} e^{-ix\xi}\right]_{-a}^{a} = -\frac{1}{i\xi}\left(e^{-ia\xi} - e^{ia\xi}\right)$$

$$= \underbrace{\frac{2}{\xi}\sin(a\xi)}_{\to 2a \text{ då } \xi \to 0}, \quad \xi \neq 0$$

$$\Rightarrow \hat{f}(0) = 2a, \quad \hat{f}(\xi) = 2\,\frac{\sin(a\xi)}{\xi}$$

Sats 7.5

Antag $f \in L^1(\mathbb{R})$

a) $\mathcal{F} f(x-a) = e^{-ia\xi}\hat{f}(x) \qquad a \in \mathbb{R}$

$\mathcal{F}\left(e^{iax} f(x)\right) = \hat{f}(\xi - a)$

b) $\widehat{f_\delta}(\xi) = \hat{f}(\xi\delta) \qquad \left\{ f_\delta(x) = \frac{1}{\delta} f\left(\frac{x}{\delta}\right) \right\}$

$\mathcal{F} f(\delta x) = (\hat{f})_\delta(\xi)$

c) om $f' \in L^1$ och f är styckvis glatt och kontinuerlig så är:

$$\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$$

om $x f(x) \in L^1$ så är

$\mathcal{F}(x f(x)) = i(\hat{f})'$ dvs $i \frac{d}{d\xi}\hat{f}(\xi) = \widehat{x f(x)}(\xi)$

d) om $f, g \in L^1$ så: $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

Bevis

(a) $$\mathcal{F} f(x-a)(\xi) = \int f(x-a) e^{-ia\xi}\, dx = \{x-a = x'\} =$$

$$= \int f(x') e^{-ix'\xi} e^{ia\xi}\, dx' = e^{-ia\xi}\hat{f}(\xi)$$

$$\cancel{\mathcal{F}(x)} \quad \mathcal{F}(e^{ia\xi} f(x))(\xi) = \int f(x) \underbrace{e^{iax} e^{-ix\xi}}_{e^{-ix(\xi-a)}} dx = \hat{f}(\xi - a)$$

(b) $$\hat{f_\delta}(\xi) = \int \frac{1}{\delta} f\left(\frac{x}{\delta}\right) e^{-ix\xi}\, dx = \hat{f}(\delta\xi)$$

(c) $$\hat{f'}(\xi) = \int f'(x) e^{-ix\xi}\, dx = \left[f(x) e^{-ix\xi}\right]_{-\infty}^{+\infty} + i\xi \int f(x) e^{-ix\xi}\, dx$$

$$\frac{d}{d\xi}\hat{f}(\xi) = \frac{d}{d\xi}\int f(x) e^{-ix\xi}\, dx \underset{\text{tillåtet}}{=} \int (-ix) f(x) e^{-ix\xi}\, dx = -i\,\widehat{xf(x)}(\xi)$$

(d) $$\widehat{f*g}(\xi) = \iint f(x-y) g(y)\, dy\, e^{-ix\xi}\, dx =$$

$$= \int \underbrace{\int f(x-y) e^{-i(x-y)\xi}\, dx}_{=\hat{f}(\xi)}\, e^{-iy\xi} g(y)\, dy = \hat{f}(\xi)\hat{g}(\xi)$$

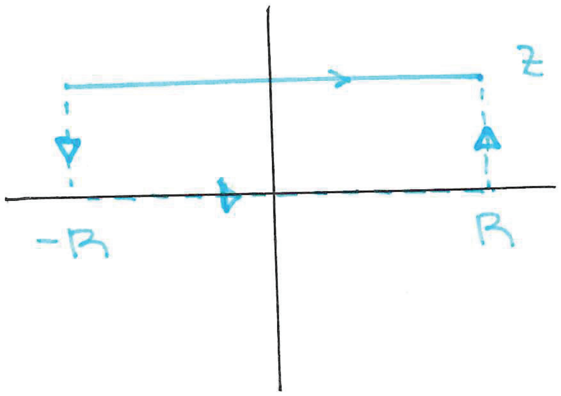
Ex. $f(x) = e^{-\frac{ax^2}{2}}$, $a > 0$ (övn. 7.2.1)

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-\frac{ax^2}{2} - ix\xi}\, dx = \int e^{-\frac{a}{2}\left(x + \frac{i\xi}{a}\right)^2 - \frac{\xi^2}{2a}}\, dx =$$

$$= \left\{ z = x + \frac{i\xi}{a} \right\} = \int e^{-\frac{a}{2}z^2}\, dz \cdot e^{-\frac{\xi^2}{2a}} =$$

$$= \lim_{R\to\infty} \int_{-R+\frac{i\xi}{a}}^{+R+\frac{i\xi}{a}} \exp\left(-\frac{a}{2}z^2\right)dz \cdot \exp\left(-\frac{\xi^2}{2a}\right) \quad \ast$$

$$\ast \int_{-R+\frac{i\xi}{a}}^{+R+\frac{i\xi}{a}} \exp\left(-\frac{a}{2}z^2\right)dz = \int_{\sqcup} \exp\left(-\frac{a}{2}z^2\right)dz$$



$$\int_{-R}^{R} \exp\left(-\frac{a}{2}z^2\right)dz \to \int_{-\infty}^{+\infty} \exp\left(-\frac{a}{2}x^2\right)dx$$

$$= \sqrt{\frac{2\pi}{a}}$$

$z = \pm R + iy$ ger $\left|\exp\left(-\frac{a}{2}z^2\right)\right| = \exp\left(-\frac{a}{2}\operatorname{Re} z^2\right) =$

$= \exp\left(-\frac{a}{2}(R^2-y^2)\right) \to 0$ då $R \to \infty$

$\{ y$ mellan 0 och $\frac{\xi}{a} \}$

$$\hat{f}(\xi) = \lim_{R\to\infty} \ldots = \exp\left(-\frac{\xi^2}{2a}\right) \cdot \sqrt{\frac{2\pi}{a}}$$