Lågpassfilter

Tors LV3

~~Ex~~.

Definiera ett tidsinvariant linjärt dynamiskt system

$S = LP_\alpha$, $\alpha > 0$ genom att systemfunktionen ges av

$\hat{h} = \chi_{(-\alpha,\alpha)}$ så att $LP_\alpha f = \mathcal{F}^{-1}(\chi_{(-\alpha,\alpha)}\hat{f})$

$$LP_\alpha f(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \hat{f}(\xi)e^{it\xi}\,d\xi$$

Impulssvaret $h(t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} e^{it\xi}\,d\xi = \frac{\sin \alpha t}{\pi t}$

$$LP_\alpha \delta(t) = \frac{\sin \alpha t}{\pi t}$$

[α kallas bandbredd!]

$$LP_\alpha \delta(t-a) = \frac{\sin \alpha (t-a)}{\pi (t-a)}$$

Obs! Ej kausalt, ej stabilt ty $\int_{-\infty}^{+\infty} \frac{\sin \alpha t}{\pi t}\,dt$ div.

Samplingssatsen

$f \in L^2(\mathbb{R})$ signal, avläses i punkter $t = nT$ där $T > 0$.
Den samplade signalen är följden $(f(nT))_{n=-\infty}^{+\infty}$

eller $\sum_{n=-\infty}^{+\infty} f(nT)\underbrace{\delta(t-nT)}_{\delta_{nT}}$

Sats

Om $f \in L^2(\mathbb{R})$ och $\hat{f}(\omega) = 0$ för $|\omega| \geq \alpha$ och $T\alpha \leq \pi$, så är f bestämd av den samplade signalen, mer precis är

$$f(t) = T \sum_{n=-\infty}^{+\infty} f(nT) \frac{\sin \alpha (t - nT)}{\pi (t - nT)}$$

Anm. (1)

Följand väljer $\alpha T = \pi$ och

$$f(t) = \sum f\left(n \frac{\pi}{\alpha}\right) \frac{\sin(\alpha t - \pi n)}{\alpha t - \pi n}$$

Anm. (2)

Kan skriva högerledet som

$$T \sum_{-\infty}^{+\infty} f(nT) LP_\alpha \delta(t - nT) = T LP_\alpha \sum \overbrace{f(nT)\delta(t-nT)}^{= f(t)\delta(t-nT)^{*}}$$

ty LP_α är linjär!

* $f(x)\delta(x-a) = f(a)\delta(x-a)$

Bevis

$$f(t) = T LP_\alpha \sum_{n=-\infty}^{+\infty} f(t)\delta(t - nT)$$ (Räcker att visa detta)

Men! $$LP_\alpha \sum_{n=-\infty}^{+\infty} f(t)\delta(t-nT) = LP_\alpha f(t) \sum_{-\infty}^{+\infty} \delta(t - nT)$$

$$= \mathcal{F}^{-1} \sum_{-\infty}^{+\infty} \chi_{(-\alpha,\alpha)} \widehat{f(t)\,\delta(t-nT)} =$$

$$= \mathcal{F}^{-1} \chi_{(-\alpha,\alpha)} \widehat{f(t) \sum_{-\infty}^{+\infty} \delta(t-nT)}$$

$\sum_{-\infty}^{+\infty} \delta(t-nT)$ är en T-periodisk "funktion" som kan Fourierserieutvecklas! $\left\{ T = 2L,\ \frac{\pi}{L} = \frac{2\pi}{T} \right\}$

$$\sum_{n=-\infty}^{+\infty} \delta(t-nT) = \sum_{k=-\infty}^{+\infty} c_k \exp\left(ik \frac{2\pi}{T} t\right)$$

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{-\infty}^{+\infty} \delta(t-nT) \exp\left(-ik\frac{2\pi}{T}t\right) dt = \frac{1}{T}$$

$$\sum_{-\infty}^{+\infty} \delta(t-nT) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} \exp\left(ik\frac{2\pi}{T}t\right)$$

Får $\mathcal{F}^{-1} \chi_{(-\alpha,\alpha)} \widehat{f(t) \sum_{-\infty}^{+\infty} \delta(t-nT)} =$

$$= \mathcal{F}^{-1} \chi_{(-\alpha,\alpha)} \frac{1}{T} \sum_{k=-\infty}^{+\infty} \widehat{f(t) \exp\left(ik\frac{2\pi}{T}t\right)} =$$

$$= \frac{1}{T} \mathcal{F}^{-1} \chi_{(-\alpha,\alpha)} \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2\pi}{T}k\right) \qquad \left\{ \frac{2\pi}{T} \geq 2\alpha \right\}$$

$$\chi(\omega) \sum \hat{f}\left(\omega - \frac{2\pi}{T}k\right) = \hat{f}(\omega)$$

$\boxed{2}$

Laplacetransformen

$$\mathcal{L} f(z) = \int_0^\infty f(t) e^{-zt} \, dt \quad , \qquad z \in \mathbb{C}$$

f definierad på $\{x : x \geq 0\}$, fortsätts med 0 i $\{x < 0\}$

om $|f(t)| \leq C e^{at}$, ngt $a > 0$ så konvergerar

$\mathcal{L} f(z)$ för $\operatorname{Re}(z) > a$

Räkneregler:

$$\mathcal{L} 1 = \frac{1}{z} \qquad\qquad \mathcal{L} \delta = 1$$

$$\mathcal{L} f'(z) = z \mathcal{L} f(z) - f(0) \Longleftrightarrow \mathcal{L}\left(f' + f(0)\delta\right) = z \mathcal{L} f(z)$$

$$\mathcal{L}\left(\int_0^t f\right)(z) = \frac{1}{z} \mathcal{L} f(z)$$

$$\mathcal{L}(f * g)(z) = \mathcal{L} f(z) \, \mathcal{L}(g(z)) \quad \text{där}$$

$$f * g(t) = \int f(t-s) g(s) \, ds = \int_0^t f(t-s) g(s) \, ds$$

$$\mathcal{L}\left(e^{-ct} f(t)\right)(z) = \mathcal{L} f(z-c)$$

$$\mathcal{L}\left(f(t-a)\right)(z) = e^{-az} \mathcal{L} f(z)$$

Obs! $\mathcal{L} f(b+i\omega) = \int \underbrace{f(t) e^{-bt}}_{\underset{\text{def.}}{=} g(t)} e^{-i\omega t} \, dt = \hat{g}(\omega)$

Inversionsformeln!

$$f(t) = e^{bt} g(t) = e^{bt} \cdot \frac{1}{2\pi} \int \hat{g}(\omega) e^{i\omega t} \, d\omega = \{ b + i\omega = z \}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{L} f(b+i\omega) e^{(b+i\omega)t} \, d\omega = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \mathcal{L} f(z) e^{zt} \, dz$$

Ex. vågekvationen

$u_{tt} = c^2 u_{xx}$, $0 < x < L$ och $t > 0$

sätt $\mathcal{U}(x,z) = \mathcal{L} u(x,z)$ Laplacetransf. i t-variabeln

$$\mathcal{L} u_{tt}(x,z) = z \mathcal{L} u_t(x,z) - u_t(x,0) =$$

$$= z^2 \mathcal{L} u(x,z) - z u(x,0) - u_t(x,0)$$

$$\mathcal{L} u_{tt}(x,z) = z^2 \mathcal{U}(x,z) \ , \ \mathcal{L} u_{xx}(x,z) = \mathcal{U}_{xx}(x,z)$$

$$\begin{cases} \mathcal{U}(0,z) = \mathcal{L} f(z) \\ \mathcal{U}(L,z) = 0 \end{cases}$$

Ekv. blir $z^2 \mathcal{U}(x,z) = c^2 \mathcal{U}_{xx}(x,z)$

Fixera z!

$$U(x,z) = a e^{\frac{z}{c}x} + b e^{-\frac{z}{c}x} \quad \left\{ a, b \text{ beror av } z \right\}$$

eller $$U(x,z) = a' e^{\frac{z}{c}(x-L)} + b' e^{-\frac{z}{c}(x-L)}$$

eller $$U(x,z) = a'' \cosh \frac{z}{c}(x-L) + b'' \sinh \frac{z}{c}(x-L)$$

$U(L,z) = 0$ ger $a'' = 0$

$U(0,z) = \mathcal{L} f(z)$ ger $-b'' \sinh \frac{z}{c} L = \mathcal{L} f(z)$

$$b'' = \frac{-\mathcal{L} f(z)}{\sinh \frac{z}{c} L}$$

$$\longrightarrow U(x,z) = \frac{\mathcal{L} f(z) \sinh \frac{z}{c}(L-x)}{\sinh \frac{z}{c} L}$$

$$\frac{1}{\sinh s} = \frac{2}{e^{s} - e^{-s}} = \frac{1}{1-e^{-2s}} \frac{2}{e^{s}} =$$

$$= \frac{2}{e^{s}} \sum_{n=0}^{\infty} e^{-2ns}$$

$$\frac{\sinh \frac{z}{c}(L-x)}{\sinh \frac{z}{c} L} = \exp\left[\left(\frac{z}{c}(L-x)\right) - \frac{z}{c} L\right] - \exp\left(-\frac{z}{c}(L-x) - \frac{z}{c} L\right) \cdot$$

$$\cdot \sum_{n=0}^{\infty} \exp(-2n \frac{z}{c} L$$

$$= \sum_{n=0}^{\infty} \exp\left[-(2nL+x)\frac{z}{c}\right] - \sum_{n=1}^{\infty} \exp\left[-(2nL-x)\frac{z}{c}\right]$$

Fortsättning följer.....