Besselfunktioner (forts.)

Mån LV6

$$r^2R'' + rR' + (\mu^2r^2 - \nu^2)R = 0 \qquad (*_\mu)$$

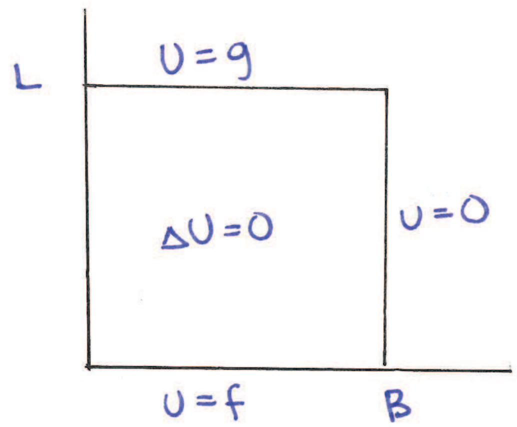
Bessel $\longrightarrow$ $R(r) = a J_\nu(\mu r) + b Y_\nu(\mu r)$

$$r^2R'' + rR' - (\mu^2r^2 + \nu^2)R = 0 \qquad (\overset{*}{*}_\mu)$$

Modif. Bessel $\longrightarrow$ $R(r) = a I_\nu(\mu r) + b K_\nu(\mu r)$

Forts. från LV5 (fredag) - cylinderexempel.

(2)



$u = u(r,z)$

$u(r,0) = f(r)$

$u(r,L) = g(r)$

$u(b,z) = 0$

$u = R(r)Z(z)$ variabelsep.

som i problem 1 (nu gör vi problem 2)

$$\frac{R'' + \frac{1}{r}R}{R} = \frac{-Z''}{Z} = \lambda \quad \text{Konstant}$$

$$r^2R'' + rR' - \lambda r^2 R = 0 \qquad \{R(b) = 0\}$$

$\lambda > 0$

sätt $\lambda = \mu^2$, $\mu > 0$; Får Bessels modifierade ekv. med $\nu = 0$

$R(r) = a I_0(\mu r) + b' K_0(\mu r)$

$b' = 0$ ty K_0 singulär i 0

Får $I_0(\mu b) = 0$ Ingen lösning!

$\underline{\lambda = 0}$

$r^2 R'' + rR' = 0$ $\left\{ \text{Eulerekv.} \quad R(r) = a + b' \ln r \right\}$

$b' = 0$ pga $r = 0$, sen $a = 0$ Ingen lösning!

$\underline{\lambda < 0}$

sätt $\lambda = -\mu^2$, $\mu > 0$; får Bessels ekv. med $\nu = 0$

$R(r) = a J_0(\mu r) + b' Y_0(\mu r)$

$b' = 0$ ty $r = 0$. RV ger $J_0(\mu b) = 0$ så att $\mu b = \lambda_k$

där $(\lambda_k)_1^\infty$ är nollställe till J_0 i $\mathbb{R}_+$

$$\mu = \frac{\lambda_k}{b} \longrightarrow \lambda = -\left(\frac{\lambda_k}{b}\right)^2$$

$$Z'' = \left(\frac{\lambda_k}{b}\right)^2 Z$$

$$Z = a \exp\left(\frac{\lambda_k}{b} z\right) + b' \exp\left(-\frac{\lambda_k}{b} z\right)$$

eller ekvivalent $Z = a'' \sinh \frac{\lambda_k}{b} z + b'' \sinh \frac{\lambda_k}{b}(L - z)$

Ansätt $u(r,z) = \sum_{k=1}^{\infty} J_0\left(\frac{\lambda_k}{b} r\right)\left(a_k \sinh \frac{\lambda_k}{b} z + b_k \sinh \frac{\lambda_k}{b}(L - z)\right)$

Randvillkor f, g ger $\sum_{k=1}^{\infty} b_k J_0\left(\frac{\lambda_k}{b} r\right) \sinh \frac{\lambda_k}{b} L = f(r)$

Gäller för $z = 0$, $0 < r < b$

Utveckla $\quad f(r) = \sum_{k=1}^{\infty} c_k J_0\left(\frac{\lambda_k}{b} r\right)$

där $\quad c_k = \dfrac{1}{\|J_0(\frac{\lambda_k}{b} r)\|_w^2} \displaystyle\int_0^b f(r) J_0\left(\frac{\lambda_k}{b} r\right) r\, dr$

där $\quad \|J_0(\frac{\lambda_k}{b} r)\|_w^2 = \frac{b^2}{2} J_1(\lambda_k)^2$

$$b_k = \frac{c_k}{\sinh \frac{\lambda_k}{b} L}$$

Analogt: $\cancel{z} = L \longrightarrow \sum a_k J_0\left(\frac{\lambda_k}{b} r\right) \sinh \frac{\lambda_k}{b} L = g(r)$

utveckla $\quad g(r) = \sum_1^{\infty} d_k J_0\left(\frac{\lambda_k}{b} r\right)$

$$a_k = \frac{d_k}{\sinh \frac{\lambda_k}{b} L}$$

klart!

Ex.

Vågekvationen i cirkelskiva (membran, trumskinn)

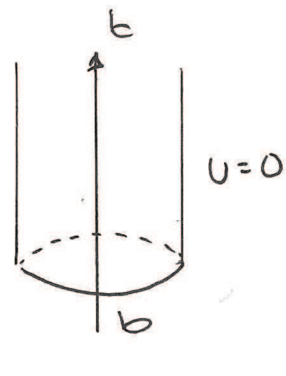
$u_{tt} = c^2 \Delta u$

$u = u(r, \theta, t) \qquad r < b, \quad t \geq 0$

$u = u(b, \theta, t) = 0$

$u(r, \theta, 0) = f(r, \theta)$

$u_t(r, \theta, 0) = 0$



separera: $\quad u = R(r)\,\Theta(\theta)\,T(t)$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda \quad \text{Konst.}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \lambda = -\frac{1}{r^2}\frac{\Theta''}{\Theta}$$

$$\frac{r^2R'' + rR'}{R} - \lambda r^2 = -\frac{\Theta''}{\Theta} = \lambda' \quad \text{Konst.}$$

$$\lambda' = n^2, \quad n = \{0,1,2,\dots\}$$

$$\Theta(\theta) = a\cos n\theta + b'\sin n\theta$$

För $r^2R'' + rR - (\lambda r^2 + n^2)R = 0$

$\lambda > 0$

$\lambda = \mu^2,\ \mu > 0$ ger Bessels. modif. ekv.

$$R(r) = c\,I_n(\mu r) + d\,K_n(\mu r)$$

$d = 0$ ty K_n singulär i 0

RV för $r = b$ ger $R(b) = 0$

Men! $I_n(\mu r) > 0 \quad \forall\, r > 0$

Ingen lösninge

$\lambda = 0$

$$r^2R'' + rR' - n^2R = 0$$

Euler ekv, ansätt $R = r^\gamma$

$$r(r-1) + \gamma - n^2 = 0$$

$r = \pm n$

$n \neq 0$: $R(r) = cr^n + dr^{-n}$

$d = 0$ ty $r = 0$ $\quad r^n > 0$ för $r = b$ Ingen lösn.

$n = 0$: $R(r) = c + d \ln r$

$d = 0$, $c = 0$

Ingen lösning

$\underline{\lambda < 0}$

Sätt $\lambda = -\mu^2$, $\mu > 0$

$r^2 R'' + rR' + (\mu^2 r^2 - n^2) R = 0$

Besselekv. ; $\nu = n$

Lösning: $c J_n(\mu r) + d\, Y_n(\mu r)$

$d = 0$ ty $r = 0$, $\longrightarrow J_n(\mu b) = 0$

$\mu b = \lambda_{kn}$ för ngt k där $(\lambda_{kn})_1^\infty$ är de positiva nollställena till J_n

$$\mu = \frac{\lambda_{kn}}{b} \longrightarrow \lambda = -\left(\frac{\lambda_{kn}}{b}\right)^2$$

$$\longrightarrow R(r) = J_n\left(\frac{\lambda_{kn}}{b} r\right)$$

T : $T''(t) = c^2 \lambda T(t)$ dvs. $T'' = -c^2 \left(\frac{\lambda_{kn}}{b}\right)^2 T$

$$T(t) = \tilde{a} \cos c \frac{\lambda_{kn}}{b} t + \tilde{b} \sin c \frac{\lambda_{kn}}{b} t$$

$T'(0) = 0$ enligt begynnelsevillkor

Men $T' = -\tilde{a} c \frac{\lambda_{kn}}{b} \sin c \frac{\lambda_{kn}}{b} t + \tilde{b} c \frac{\lambda_{kn}}{b} \cos c \frac{\lambda_{kn}}{b} t$

$\Rightarrow \tilde{b} = 0 \Rightarrow T(t) = \cos c \frac{\lambda_{kn}}{b} t$

Sep. lösning. $J\left(\frac{\lambda_{kn}}{b} r\right)(a \cos n\theta + b' \sin n\theta) \cos \frac{c\lambda_{kn}}{b} t$

Ansätt $u(r,\theta,t) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} J_n\left(\frac{\lambda_{kn}}{b} r\right)\left(a_{kn} \cos n\theta + b_{kn} \sin n\theta\right) \cos \frac{c\lambda_{kn}}{b} t$

$t = 0$ skall ge $u(r,\theta,0) = f(r,\theta)$

$\Rightarrow \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} J_n\left(\frac{\lambda_{kn}}{b} r\right)\left(a_{kn} \cos n\theta + b_{kn} \sin \theta n\right) = f(r,\theta)$

Sats 5.4

Funktionerna

$(r,\theta) \mapsto J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos n\theta \qquad k = 1, 2, \ldots, \quad n = 0, 1, \ldots$

och

$(r,\theta) \mapsto J_n\left(\frac{\lambda_{kn}}{b} r\right) \sin n\theta \qquad k = 1, 2, \ldots, \quad n = 0, 1, \ldots$

bildar tillsammans ett fullständigt ortogonalsystem

$L^2_w = L^2\left([0,b] \times [-\pi, \pi] \, ; \, r \, dr \, d\theta\right)$

dvs. $L^2(D)$, $D = \{(x,y) : x^2 + y^2 \leq b^2\}$ med $\iint \ldots dx\,dy$

$\square$

Med denna sats: a_{kn}, b_{kn} är då koeff. för f i detta ortogonal-system, alltså

$$a_{kn} = \frac{1}{\left\| J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos n\theta \right\|_w^2} \int_0^b \int_{-\pi}^{\pi} f(r,\theta) \, J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos n\theta \, r \, dr \, d\theta$$

$$\left\| J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos n\theta \right\|_w^2 = \iint J_n\left(\frac{\lambda_{kn}}{b} r\right)^2 \cos^2 n\theta \; r\,dr\,d\theta =$$

$$= \underbrace{\int_0^b J_n\left(\frac{\lambda_{kn}}{b} r\right)^2 r\,dr}_{\frac{b^2}{2} J_{n+1}(\lambda_{kn})^2} \; \underbrace{\int_{-\pi}^{\pi} \cos^2 n\theta \, d\theta}_{\begin{cases} \pi & n>0 \\ 2\pi & n=0 \end{cases}}$$

b_{kn} fås analogt!

Klart!

Bevis (sats 5.4)

Ortog., ta $(k,n) \neq (k',n')$

$$\iint J_n\left(\frac{\lambda_{kn}}{b} r\right) \cos n\theta \; J_{n'}\left(\frac{\lambda_{k'n'}}{b} r\right) \cos n'\theta \; r\,dr\,d\theta$$

om $n \neq n'$: $\int \cos n\theta \cos n'\theta \, d\theta = 0$

om $n = n'$, $k \neq k'$: $\int J_n\left(\frac{\lambda_{kn}}{b} r\right) J_{n'}\left(\frac{\lambda_{k'n'}}{b} r\right) r\,dr = 0$