Associerade Legendre fkner

{Mån Lv7}

$$P_n^m(s) = (1-s^2)^{\frac{m}{2}} \frac{d^m}{ds^m} P_n(s), \quad 0 \leq m \leq n$$

$(P_n^m)_{n=m}^{\infty}$ bildar $\forall m$ ett fullständigt ortogonal-system. i $L^2[-1,1]$

$P_n^0 = P_n$ legendre-polynom

$\Delta U = 0$ i $r < R_0$ $\quad U = U(r, \theta, \varphi)$ {sfäriska koord.}

Sep. lösning. $R(r)\,\Theta(\theta)\,\Phi(\varphi)$

Får: $\underbrace{r^n e^{im\varphi} P_n^{(m)}(\cos\theta)}$ $\quad m \in \mathbb{Z}, \ n \geq |m|$

klotytefunktioner $Y_{mn}(\theta, \varphi)$ (spherical harmonics) } bildar fullständigt ortogonalsystem!

Bildar fullständigt ortogonalsystem i

$$L^2([0,\pi] \times [-\pi, \pi] ; \sin\theta \, d\theta \, d\varphi)$$

dvs på enhetssfären med ytmåttet.

Ansätt $$U(r, \theta, \varphi) = \sum_{m=-\infty}^{+\infty} \sum_{n=|m|}^{\infty} c_{mn} r^n e^{im\varphi} P_n^{|m|}(\cos\theta)$$

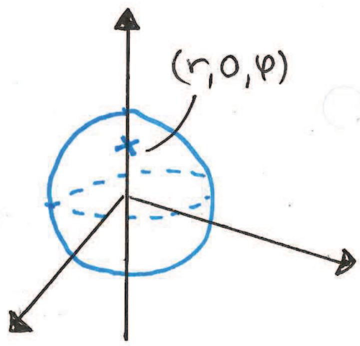
specialfall: oberoende av φ: $\sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)$

Givet randvillkor $\quad u(R_0, \theta, \varphi) = f(\theta, \varphi)$

fås $\displaystyle\sum_{m=-\infty}^{+\infty} \sum_{n=|m|}^{\infty} c_{mn} R_0^n e^{im\varphi} P_n^{|m|}(\cos\theta) = f(\theta,\varphi)$

där $\displaystyle c_{mn} = R_0^{-n} \frac{1}{\underbrace{\| e^{im\varphi} P_n^{|m|}(\cos\theta)\|^2_{\sin\theta}}_{\sigma}} \iint f(\theta,\varphi) e^{im\varphi} P_n^{|m|}(\cos\theta)\sin\theta\, d\theta\, d\varphi$

$$\|\sigma\|^2_{\sin\theta} = \underbrace{\|e^{im\varphi}\|^2}_{2\pi} \underbrace{\|P_n^{|m|}(\cos\theta)\|^2_{\sin\theta}}_{\text{känt}}$$



Anta $R_0 = 1$, beräkna $u(r, \theta, \varphi)$ för $0 < r < 1$

$$\Rightarrow u(r,0,\varphi) = \sum\sum c_{mn} r^n e^{im\varphi} \underbrace{P_n^{|m|}(1)}_{\begin{cases} =0 & \text{för } |m|>0 \\ =1 & \text{för } |m|=0 \end{cases}} =$$

enligt def.

$$= \sum_{n=0}^{\infty} c_n r^n$$

$$c_n = \frac{1}{\underbrace{2\pi \|P_n(\cos\theta')\|^2_{\sin\theta}}_{\frac{1}{\frac{4\pi}{2n+1}} = \frac{2n+1}{4\pi}}} \iint f(\theta',\varphi') P_n(\cos\theta')\sin\theta'\, d\theta'\, d\varphi'$$

$$u(r,0,\varphi) = \frac{1}{4\pi} \iint f(\theta',\varphi') \sum_{n=0}^{\infty} \underbrace{(2n+1) r^n}_{\left(2r\frac{d}{dr}+1\right)} P_n(\cos\theta')\sin\theta'\, d\theta'\, d\varphi'$$

$=$

$$\sum_0^\infty r^n P_n(\cos\theta') = \frac{1}{(1+r^2-2r\cos\theta')^{\frac{1}{2}}} \qquad 0 \leq r < 1$$

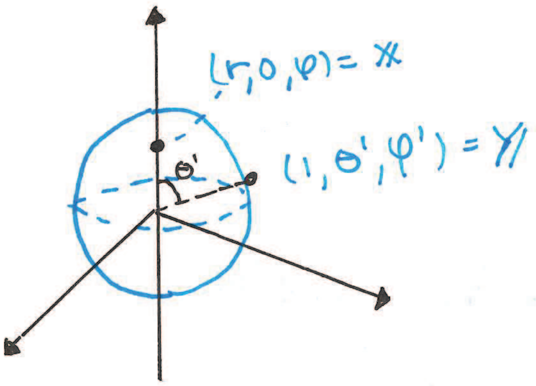
$$= \frac{1}{4\pi}\iint f(\theta',\varphi')\left(2r\frac{d}{dr}+1\right)\underbrace{\sum_0^\infty r^n P_n(\cos\theta')}_{\frac{1}{(1+r^2-2r\cos\theta)^{\frac{1}{2}}}}\sin\theta'\, d\theta'\, d\varphi' =$$

$$= \iint f(\theta',\varphi')\,\rho(r,\theta',\varphi')\sin\theta'\, d\theta'\, d\varphi'$$

där $\rho(r,\theta',\varphi') = \frac{1}{4\pi}\left(2r\frac{d}{dr}+1\right)\frac{1}{(1+r^2-2r\cos\theta')^{\frac{1}{2}}} =$

$$= \frac{1}{4\pi}\,\frac{1-r^2}{(\underbrace{1-r^2-2r\cos\theta'}_{|X-Y|^2})^{\frac{3}{2}}}$$

cos-teoremet



$$\rho(r,\theta',\varphi') = \frac{1}{4\pi}\,\frac{1-|X|^2}{|X-Y|^3}$$

Rotera. För alla X fås:

$$u(X) = \frac{1}{4\pi}\int_{|Y|=1} f(Y)\,\frac{1-|X|^2}{|X-Y|^3}\,dS(Y)$$

Poissonintegralen!

Allmänt R_0:

$$u(X) = \frac{1}{4\pi R_0}\int_{\|Y\|=R_0} f(Y)\,\frac{R_0^2-|X|^2}{|X-Y|^3}\,dS(Y)$$

Laguerre-polynom

Vikt $w(x) = x^{\alpha} e^{-x}$ i $(0, +\infty)$, $\alpha > -1$

Polynom $(L_n^{\alpha})_{n=0}^{\infty}$ def av.

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^{x} \left(\frac{d}{dn}\right)^n \left(x^{\alpha+n} e^{-x}\right) \quad \left\{ \begin{array}{c} \text{Rodrigues} \\ \text{formel} \end{array} \right\}$$

Fullständigt ortogonal-system i $L^2_{w(x)}[0, \infty]$ för varje fixt α

Diff. ekv. $(x^{\alpha+1} e^{-x} f')' + \lambda x^{\alpha} e^{-x} f = 0$

har $\lambda = n \in \{0, 1, \ldots\}$ som egenvärden och L_n^{α} som egenfunktioner

Klotytefunktioner

$y_{mn} = e^{im\varphi} P_n^{|m|}(\cos\theta)$ $m \in \mathbb{Z}$, $n \geq |m|$

$r^n y_{mn}(\theta, \varphi)$ är harmonisk funktion dvs $\Delta \ldots = 0$

$P_0(s) = 1$ $P_1(s) = s$ $P_2(s) = \frac{3}{2}s^3 - \frac{1}{2}$

m=0 $\Rightarrow$ $y_{00}(\theta, \varphi) = 1$ $y_{01}(\theta, \varphi) = \cos\theta = r\cos\theta = z$

$$y_{02}(\theta, \varphi) = \frac{3}{2}\cos^2\theta - \frac{1}{2} = \frac{3}{2}r^2\cos^2\theta - \frac{1}{2}r^2 =$$

$$= \frac{3}{2}z^2 - \frac{1}{2}(x^2 + y^2 + z^2) = z^2 - \frac{x^2 + y^2}{2}$$

$\underline{m = \pm 1}$ $\quad Y_{\pm 11}(\theta, \varphi) = e^{\pm i\varphi}(1-s^2)^{\frac{1}{2}} \frac{d}{ds} s = \{ s = \cos\theta \}$

$$= \sin\theta \cdot \begin{cases} \cos\varphi \\ \sin\varphi \end{cases} = \begin{cases} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \end{cases} = \begin{cases} x \\ y \end{cases}$$

$$Y_{\pm 12} = e^{\pm 2i\varphi}(1-s^2)^{\frac{1}{2}} \frac{d}{ds}\left(\frac{3}{2}s^2 - \frac{1}{2}\right) = 3\sin\theta\cos\theta \begin{cases} \cos\varphi \\ \sin\varphi \end{cases}$$

$$= 3 \begin{cases} xz \\ yz \end{cases}$$

$m = \pm 2$ $\quad Y_{\pm 22} = e^{\pm 2i\varphi}(1-s^2) \frac{d^2}{ds^2}\left(\frac{3s^2}{2} - \frac{1}{2}\right) = 3\sin^2\theta \begin{cases} \cos 2\varphi \\ \sin 2\varphi \end{cases} =$

$$= \overset{r^2}{\underset{\wedge}{3}} \begin{cases} \sin^2\theta(\cos^2\varphi - \sin^2\varphi) \\ \sin^2\theta \cdot 2\sin\varphi\cos\varphi \end{cases} =$$

$$= \begin{cases} 3(x^2 - y^2) \\ 6xy \end{cases}$$