4.11.h) $h(x) = x^2$

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) f(x)\,dx = \int_0^2 \frac{x^3}{2}\,dx = \left[\frac{x^4}{8}\right]_0^2 = \underline{\underline{2}}$$

4.13

$$f(x) = \begin{cases} \frac{k}{x^4} & x > 1 \\ 0 & x \leq 1 \end{cases}$$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_1^{\infty} \frac{k}{x^4}\,dx = -\left[\frac{k}{3x^3}\right]_1^{\infty} = F(\infty) - F(1) = \frac{k}{3} = 1 \Rightarrow \underline{\underline{k = 3}}$

$\Rightarrow f(x) = \frac{3}{x^4}$

b) $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{x} \frac{k}{y^4}\,dy = -\left[y^{-3}\right]_1^x = \underline{\underline{-x^{-3} + 1}}$

c) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 + 2^{-3} - 1 = \underline{\underline{0{,}125}}$

$P(2 < X < 3) = F(3) - F(2) = -3^{-3} + 1 + 2^{-3} - 1 \approx \underline{\underline{0{,}0880}}$

d) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_1^{\infty} \frac{3}{x^3}\,dx = -\left[\frac{3}{2x^2}\right]_1^{\infty} = \frac{3}{2} = \underline{\underline{1{,}5}} = \mu_X$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_1^{\infty} (x - 1{,}5)^2 \frac{3}{x^4}\,dx = \int_1^{\infty} \left(\frac{3}{x^2} - \frac{6 \cdot 1{,}5}{x^3} + \frac{1{,}5^2}{x^4}\right) dx =$$

$$= \left[-3x^{-1} + 3 \cdot 1{,}5 x^{-2} - \frac{3 \cdot 1{,}5^2}{3} x^{-3}\right]_1^{\infty} = 3 - 4{,}5 + \cancel{3}\frac{1{,}5^2}{\cancel{3}} = \underline{\underline{0{,}75}}$$

$\Rightarrow \sigma_X = \sqrt{0{,}75} \approx \underline{\underline{0{,}866}}$

e) $P(\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X) = P(0{,}634 \leq X \leq 2{,}366) = P(1 \leq X \leq 2{,}366) =$

$= F(2{,}366) = -2{,}366^{-3} + 1 \approx \underline{\underline{0{,}924}}$

4.19

$$F(x) = \begin{cases} 0 & x \leq \\ \frac{x}{4}\left(1 + \ln\frac{4}{x}\right) & 0 < x \leq 4 \\ 1 & x > 4 \end{cases}$$

a) $P(X \leq 1) = \left[\frac{x}{4}\left(1 + \ln\frac{4}{x}\right)\right]_0^1 = \frac{1}{4}(1 + \ln 4) \approx 0{,}597$

b) $P(1 \leq X \leq 3) = F(3) - F(1) = \frac{3}{4}\left(1 + \ln\frac{4}{3}\right) - \frac{1}{4}(1 + \ln 4) \approx 0{,}369$

c) $F(x) = \frac{x}{4}(1 + \ln 4 - \ln x) \Rightarrow f(x) = F'(x) = \frac{1}{4}(1 + \ln 4 - \ln x) +$

$+ \frac{x}{4}\left(-\frac{1}{x}\right) = \cancel{\frac{1}{4}} + \frac{\ln 4}{4} - \frac{\ln x}{4} - \cancel{\frac{1}{4}} = \underline{0{,}347 - 0{,}25 \ln x} = 0{,}25(\ln 4 - \ln x)$

4.23 a) $\eta(p) = A + (B-A)p$

b) $E(X) = \mu_x = \int_{-\infty}^{\infty} x P(x)\,dx = \int_b^{\infty} x \cancel{P(x)}\,dx + \int_a^b x P(x)\,dx + \int_{-\infty}^{a} x \cancel{P(x)}\,dx =$

$$= \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \underline{\underline{\frac{b+a}{2}}} = \mu_x$$

$$V(X) = \sigma_x^2 = E\left((x-\mu_x)^2\right) = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx = \frac{1}{3(b-a)}\left[\left(x - \frac{a+b}{2}\right)^3\right]_a^b =$$

$$= \frac{1}{3(b-a)}\left(\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3\right) = \frac{1}{3(b-a)}\left(\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right) =$$

$$= \frac{1}{3(b-a)}\,\frac{2(b-a)^3}{8} = \underline{\underline{\frac{(b-a)^2}{12}}}$$

$$\sigma_x = \sqrt{V(x)} = \underline{\underline{\frac{b-a}{\sqrt{12}}}}$$

c) $E(X^n) = \int_a^b x^n P(x)\,dx = \int_a^b x^n \frac{1}{b-a}\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b \frac{1}{b-a} = \underline{\underline{\frac{b^{n+1}-a^{n+1}}{(1+n)(b-a)}}}$

4.61 $\lambda = 1$

a) $\mu = \frac{1}{\lambda} = \underline{\underline{1}}$

b) $\sigma^2 = \frac{1}{\lambda^2} \Rightarrow \sigma = \frac{1}{\lambda} = \underline{\underline{1}}$

c) $P(X \le 4) = F(4;1) = \left[F(x;\lambda) = 1 - e^{-\lambda x}\right] = 1 - e^{-4} \approx \underline{\underline{0{,}982}}$

d) $P(2 \le X \le 5) = F(5;1) - F(2;1) = 1 - e^{-5} - 1 + e^{-2} \approx \underline{\underline{0{,}129}}$

4.28 a) $P(0 \le Z \le 2{,}17) = \phi(2{,}17) - \phi(0) = 0{,}9850 - 0{,}5000 = \underline{\underline{0{,}485}}$

e) $P(Z \le 1{,}37) = \underbrace{0{,}9147}_{\phi(1{,}37)} \approx \underline{\underline{0{,}915}}$

g) $P(-1{,}50 \le Z \le 2{,}00) = \phi(2{,}00) - \phi(-1{,}50) = 0{,}9772 - 0{,}0668 \approx \underline{\underline{0{,}910}}$

i) $P(1{,}50 \le Z) = 1 - \phi(1{,}50) = 1 - 0{,}9332 = \underline{\underline{0{,}0668}}$

j) $P(|Z| \le 2{,}50) = P(-2{,}50 \le Z \le 2{,}50) = \phi(2{,}50) - \phi(-2{,}50) =$

$$= 0{,}9938 - 0{,}0062 \approx \underline{\underline{0{,}988}}$$

4.29 a) $\phi(c) = 0{,}9838 \Rightarrow c = \underline{\underline{2{,}14}}$

4.29 b) $P(0 \le Z \le c) = 0{,}291 \Rightarrow \Phi(c) - \Phi(0) = 0{,}291 \Rightarrow \Phi(c) = 0{,}291 + 0{,}500 = 0{,}791$

$\Rightarrow c = \underline{0{,}81}$

e) $P(c \le |Z|) = 0{,}016 \Rightarrow P(-Z \le c \le Z) = P(Z \le -c) + P(Z \ge c) =$

$= P(Z \le -c) + 1 - P(Z < c) = \Phi(-c) + 1 - \Phi(c) = 0{,}016$

$\Rightarrow \Phi(c) - \Phi(-c) = 0{,}984 \Rightarrow -c = -2{,}41$

$\Phi(-c) = 0{,}008, \quad 1 - \Phi(c) = 0{,}008 \rightarrow c = \underline{2{,}41}$

4.35 $\mu = 8{,}8; \ \sigma = 2{,}8$

a) $P(X \ge b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right) \Rightarrow P(X \ge 10) = 1 - \Phi\left(\frac{10-8{,}8}{2{,}8}\right) = 1 - \Phi(0{,}43) =$

$= 1 - 0{,}6664 = \underline{0{,}3336}$

c) $P(5 < X < 10) = P\left(\frac{5-8{,}8}{2{,}8} \le Z \le \frac{10-8{,}8}{2{,}8}\right) = P(-1{,}36 \le Z \le 0{,}43) =$

$= \Phi(0{,}43) - \Phi(-1{,}36) = 0{,}6664 - 0{,}0869 = \underline{0{,}5795}$

d) $P(8{,}8 - c \le X \le 8{,}8 + c) = 0{,}98 \Rightarrow P\left(-\frac{c}{2{,}8} \le Z \le \frac{c}{2{,}8}\right) =$

$= \Phi\left(\frac{c}{2{,}8}\right) - \Phi\left(-\frac{c}{2{,}8}\right) = 0{,}98$

$\Rightarrow \Phi\left(\frac{c}{2{,}8}\right) - \Phi\left(-\frac{c}{2{,}8}\right) = 0{,}9901 - 0{,}0099$

$\Phi\left(\frac{c}{2{,}8}\right) = 0{,}9901 \rightarrow c = 2{,}8 \cdot 2{,}33 = 6{,}524$

$\Phi\left(-\frac{c}{2{,}8}\right) = 0{,}0099 \Rightarrow c = (-2{,}8)(-2{,}33) = \underline{6{,}524}$

4.42 $\mu = 80; \ \sigma = 10$

a) $P(X \le 100) = \Phi\left(\frac{100-80}{10}\right) = \Phi(2) = \underline{0{,}9772}$

e) $P(85 \le X \le 95) = \Phi\left(\frac{95-80}{10}\right) - \Phi\left(\frac{85-80}{10}\right) = \Phi(1{,}5) - \Phi(0{,}5) =$

$= 0{,}9332 - 0{,}6915 = 0{,}2417$ Fel i facit, de har anv. $\Phi(-0{,}5)$...

f) $P(|X-80| \le 10) = P(-10 \le X - 80 \le 10) = P(70 \le X \le 90) = \left(\frac{70-80}{10} \le Z \le \frac{90-80}{10}\right) =$

$= \Phi(1) - \Phi(-1) = 0{,}8413 - 0{,}1587 = \underline{0{,}6826}$

4.117 $Y = \sigma Z + \mu$

$Y \le y \Longrightarrow \sigma Z + \mu \le y \Longrightarrow Z \le \frac{y-\mu}{\sigma}$

5.1 a) $P(X=1 \text{ och } Y=1) = p(1,1) = 0,20$

b) $P(X \le 1 \text{ och } Y \le 1) = 0,10 + 0,04 + 0,08 + 0,20 = 0,42$

c) Både X och Y har ett värde större än noll.

$P(X \ne 0 \text{ och } Y \ne 0) = P(X > 0 \text{ och } Y > 0) = 0,20 + 0,06 + 0,14 + 0,30 = 0,70$

d) $$p_x(x) = \sum_y p(x,y) = \begin{cases} 0,10 + 0,04 + 0,02 = 0,16 & x=0 \\ 0,08 + 0,20 + 0,06 = 0,34 & x=1 \\ 0,06 + 0,14 + 0,30 = 0,50 & x=2 \end{cases}$$

$$p_y(y) = \sum_x p(x,y) = \begin{cases} 0,24 & y=0 \\ 0,38 & y=1 \\ 0,38 & y=2 \end{cases}$$

$p(X \le 1) = 0,16 + 0,34 = 0,50$

e) $p(0,0) \ne p_x(0)p_y(0) \Longrightarrow$ Nej, de är inte oberoende.

5.9 $$f(x,y) = \begin{cases} K(x^2+y^2) & 20 \le x \le 30,\ 20 \le y \le 30 \\ 0 & \text{Annars} \end{cases}$$

a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Longrightarrow \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy = K\int\left[\frac{x^3}{3} + xy^2\right]_{20}^{30} dy =$$

$$= K\int_{20}^{30}\left(\frac{30^3-20^3}{3} + 10y^2\right)dy = K\left[\frac{30^3-20^3}{3}y + 10\frac{y^3}{3}\right]_{20}^{30} = K\left(\frac{30^3-20^3}{3}10 + \frac{10}{3}(30^3-20^3)\right) =$$

$$= \frac{20}{3}(30^3-20^3)K = 1 \Longrightarrow K = \frac{3}{380\,000}$$

b) $$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{3}{380\,000}\int_{20}^{30}(x^2+y^2)\,dy = \frac{3}{380\,000}\left[x^2y + \frac{y^3}{3}\right]_{20}^{30} =$$

$$= \frac{3}{380\,000}\left(10x^2 + \frac{30^3-20^3}{3}\right) = 10Kx^2 + 0,05 \qquad 20 \le x \le 30$$

c) Nej! $f(x,y) \ne f_x(x)f_y(y)$

5.12 $$f(x,y) = \begin{cases} xe^{-x(1+y)} & x \ge 0 \text{ och } y \ge 0 \\ 0 & \text{Annars} \end{cases}$$

Lösningar skrivna av Jonas Elmwall
Lp2 2008

FORTSÄTTNING 5.12

a) $f_X(x) = \int_{-\infty}^{\infty} x e^{-x(1+y)}\,dy = \int_{0}^{\infty} x e^{-x(1+y)}\,dy = \left[-e^{-x(1+y)}\right]_{0}^{\infty} = e^{-x}$

$p(X>3) = 1 - p(X \le 3) = 1 - \int_0^3 e^{-x}\,dx = 1 - (-e^{-3}+1) = e^{-3}$

b) $f_X(x) = e^{-x}$ (se a)) $x \ge 0$

$f_Y(y) = \int_0^{\infty} x e^{-x(1+y)}\,dx = \left[-x \frac{e^{-x(1+y)}}{1+y}\right]_0^{\infty} - \int_0^{\infty}\left(\frac{-e^{-x(1+y)}}{1+y}\right)dx =$

$= 0 - \left[\frac{e^{-x(1+y)}}{(1+y)^2}\right]_0^{\infty} = \frac{1}{(1+y)^2}$ $y \ge 0$

$f_X(x) f_Y(y) \ne f(x,y) \Longrightarrow$ Inte oberoende!

c)

A: Minst en överstiger 3

$P(A) = p(X>3) + p(Y>3) = \int_3^{\infty} e^{-x}\,dx + \int_3^{\infty} \frac{1}{(1+y)^2}\,dy = \left[-e^{-x}\right]_3^{\infty} + \left[-\frac{1}{1+y}\right]_3^{\infty} =$

$= \left(0 - (-e^{-3})\right) + \left(0 - \left(-\frac{1}{1+3}\right)\right) = e^{-3} + \frac{1}{4} \approx 0.2998$

5.17

$f(x,y) = \begin{cases} \frac{1}{\pi R^2} & x^2+y^2 \le R^2 \\ 0 & \text{Annars} \end{cases}$

a) $p(r \le R/2) = p\left(A \le \pi\left(\frac{R}{2}\right)^2\right) = \int_0^{\pi\left(\frac{R}{2}\right)^2} \frac{1}{\pi R^2}\,dA = \frac{\pi\left(\frac{R}{2}\right)^2}{\pi R^2} = \frac{1}{4} = 0.25$ Bör finnas en annan lösning.

4 identiska kvartscirklar

b) $p\left(X \le \frac{R}{2} \text{ och } Y \le \frac{R}{2}\right) = \int_0^{\frac{R}{2}}\int_0^{\frac{R}{2}} \frac{4}{\pi R^2}\,dx\,dy = \int_0^{\frac{R}{2}} \frac{2}{\pi R}\,dy = \frac{1}{\pi} \approx 0.318$

c) $p\left(X \le \frac{R}{\sqrt{2}} \text{ och } Y \le \frac{R}{\sqrt{2}}\right) = \int_0^{\frac{R}{\sqrt{2}}}\int_0^{\frac{R}{\sqrt{2}}} \frac{4}{\pi R^2}\,dx\,dy = \int_0^{\frac{R}{\sqrt{2}}} \left[\frac{4x}{\pi R^2}\right]_0^{\frac{R}{\sqrt{2}}} dy = \int_0^{\frac{R}{\sqrt{2}}} \frac{2\sqrt{2}}{\pi R}\,dy =$

$= \left[\frac{2\sqrt{2}\,y}{\pi R}\right]_0^{\frac{R}{\sqrt{2}}} = \frac{2}{\pi} \approx 0.637$

d) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy = \left[\frac{y}{\pi R^2}\right]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} = \frac{\sqrt{R^2-x^2} - \left(-\sqrt{R^2-x^2}\right)}{\pi R^2} =$

$= \frac{2\sqrt{R^2-x^2}}{\pi R^2}$ $-R \le x \le R$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dx = \frac{2\sqrt{R^2-y^2}}{\pi R^2}$ $-R \le y \le R$

$f_X(x) f_Y(y) \ne f(x,y) \Longrightarrow$ Ej oberoende!